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H. WEYL
THE INSTITUTE FOR ADVANCED STUDY

F. D. MURNAGHAN
THE JOHNS HOPKINS UNIVERSITY

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A NEW KIND OF RELATIONSHIP BETWEEN MATRICES.*

By CLAUDE CHEVALLEY.

The consideration of algebraic Lie groups and of their Lie algebras has led me to the study of a new kind of relationship between matrices, which seems to be of some interest by itself. The present paper is concerned with the study of this relationship. The connection with the theory of Lie algebras will be explained in a later paper.

1. The matrices $X_{r,s}$. The notion of replica. Let X, Y be two square matrices of degrees m, n respectively. If k is any integer, we shall denote by E_k the unit matrix of degree k. We set

$$X \boldsymbol{+} Y = X \boldsymbol{\times} E_n + E_m \boldsymbol{\times} Y$$

where the sign $\boldsymbol{\times}$ is used to represent the Kronecker multiplication of matrices. The matrix $X \boldsymbol{+} Y$ is called the Kronecker sum of X and Y.

If X is any square matrix, we denote by ${}^tX$ the transpose of X and by X^* the matrix $-{}^tX$.

The Kronecker addition of matrices is obviously a distributive operation with respect to both of its arguments. It is easy to recognize that it is associative. But it is in general not commutative.

If X is any matrix, we set

$$X_{r,s} = \underbrace{X^* \boldsymbol{+} X^* \boldsymbol{+} \cdots \boldsymbol{+} X^*}_{r \text{ times}} \boldsymbol{+} \underbrace{X \boldsymbol{+} X \boldsymbol{+} \cdots \boldsymbol{+} X}_{s \text{ times}}.$$

If X is considered as an infinitesimal operation on the vectors of a certain vector space, $X_{r,s}$ indicates how X operates on the r times contravariant and s times covariant tensors.

By a vector of type (r, s) we understand a column of m^{r+s} numbers (a "number" is an element of the basic field out of which the coefficients of our matrices are selected). A vector $\vec{e}$ of type (r, s) is said to be an *invariant* of X if we have

$$X_{r,s}\vec{e} = 0.$$

DEFINITION 1. *A matrix Y is said to be a replica of X if every invariant of X is also an invariant of Y.*

* Received August 14, 1942.

Let K be our basic field. It will be convenient to make use of a vector space $\mathfrak{M}$ of dimension n over K. If a base $(\vec{e}_1, \vec{e}_2, \cdots, \vec{e}_n)$ is selected in $\mathfrak{M}$, the matrices of degree n may be considered as linear endomorphisms of $\mathfrak{M}$. We denote by $\mathfrak{M}^*$ the dual space of $\mathfrak{M}$ (i. e. the space of linear functions on $\mathfrak{M}$ with values in K). If λ_i is the linear function defined by $\lambda_i(\vec{e}_j) = \delta_{ij}$, the elements $\lambda_1, \lambda_2, \cdots, \lambda_n$ form a base of $\mathfrak{M}^*$, the dual of the base $(\vec{e}_1, \cdots, \vec{e}_n)$ of $\mathfrak{M}$. On the other hand, if $\mathfrak{N}, \mathfrak{P}$ are two vector spaces of dimensions n, p over K, we denote by $\mathfrak{N} \mathbf{X} \mathfrak{P}$ the Kronecker product of $\mathfrak{N}$ and $\mathfrak{P}$. To every pair $(\vec{f}, \vec{g})$ composed of a vector $\vec{f} \in \mathfrak{N}$ and of a vector $\vec{g} \in \mathfrak{P}$ there corresponds a vector $\vec{f} \mathbf{X} \vec{g} \in \mathfrak{N} \mathbf{X} \mathfrak{P}$; $\vec{f} \mathbf{X} \vec{g}$ is a linear function of each of its arguments when the other is kept fixed. If $(\vec{f}_1, \vec{f}_2, \cdots, \vec{f}_n)$, $(\vec{g}_1, \cdots, \vec{g}_p)$ are bases for $\mathfrak{N}, \mathfrak{P}$ respectively, the vectors $\vec{f}_i \mathbf{X} \vec{g}_j$ form a base for $\mathfrak{N} \mathbf{X} \mathfrak{P}$.

We set

$$\mathfrak{M}_{r,s} = \underbrace{\mathfrak{M}^* \mathbf{X} \mathfrak{M}^* \mathbf{X} \cdots \mathbf{X} \mathfrak{M}^*}_{r \text{ times}} \mathbf{X} \underbrace{\mathfrak{M} \mathbf{X} \mathfrak{M} \mathbf{X} \cdots \mathbf{X} \mathfrak{M}}_{s \text{ times}}.$$

If $\mu_1, \mu_2, \cdots, \mu_r$ are r elements of $\mathfrak{M}^*$ and $\vec{f}_1, \cdots, \vec{f}_s$ s elements of $\mathfrak{M}$, the element $\mu_1 \mathbf{X} \mu_2 \mathbf{X} \cdots \mathbf{X} \mu_r \mathbf{X} \vec{f}_1 \mathbf{X} \cdots \mathbf{X} \vec{f}_s$ is an element of $\mathfrak{M}_{r,s}$ (a tensor of type (r, s) !). The m^{r+s} vectors $\lambda_{i_1} \mathbf{X} \lambda_{i_2} \mathbf{X} \cdots \mathbf{X} \lambda_{i_r} \mathbf{X} \vec{e}_{j_1} \mathbf{X} \vec{e}_{j_2} \mathbf{X} \cdots \mathbf{X} \vec{e}_{j_s}$ $(1 \leq i_\alpha \leq m,\ 1 \leq j_\beta \leq m)$ form a base of $\mathfrak{M}_{r,s}$. The matrix $X_{r,s}$ defines a linear endomorphism of $\mathfrak{M}_{r,s}$; more specifically, we have

$$\begin{aligned} X_{r,s}\mu_1 \mathbf{X} \mu_2 \mathbf{X} \cdots \mathbf{X} \mu_n \mathbf{X} \vec{f}_1 \mathbf{X} \vec{f}_2 \mathbf{X} \cdots \mathbf{X} \vec{f}_s \\ = \sum_{i=1}^{r} \mu_1 \mathbf{X} \cdots \mathbf{X} X^*\mu_i \mathbf{X} \cdots \mathbf{X} \mu_r \mathbf{X} \vec{f}_1 \mathbf{X} \vec{f}_2 \mathbf{X} \cdots \mathbf{X} \vec{f}_s \\ + \sum_{j=1}^{s} \mu_1 \mathbf{X} \mu_2 \mathbf{X} \cdots \mathbf{X} \vec{\mu}_r \mathbf{X} \vec{f}_1 \mathbf{X} \cdots \mathbf{X} X\vec{f}_j \mathbf{X} \cdots \mathbf{X} \vec{f}_s \end{aligned}$$

where $X^*\mu_i$ is the linear function defined by $X^*\mu_i(\vec{e}) = -\mu_i(X\vec{e})$ for any vector $\vec{e}$.

Let us consider, in particular, the space $\mathfrak{M}^* \mathbf{X} \mathfrak{M} = \mathfrak{M}_{1,1}$. There exists a natural isomorphism I between $\mathfrak{M}_{1,1}$ and the space of linear endomorphisms of $\mathfrak{M}$, which is defined in the following way: if $\mu \in \mathfrak{M}^*$, $\vec{f} \in \mathfrak{M}$, $I(\mu \mathbf{X} f)$ is the endomorphism ϕ which is defined by $\phi(\vec{e}) = \mu(\vec{e})f$ for any

vector $\vec{e}$. From now on we shall, whenever it is convenient, identify the linear endomorphisms of $\mathfrak{M}$ (i. e. also the matrices of degree m) with the corresponding elements of $\mathfrak{M}_{1,1}$.

Let ϕ be an element of $\mathfrak{M}_{1,1}$ which is of the form $\mu \times \vec{f}$. We have

$$X_{1,1}\phi = (X^*\mu) \times \vec{f} + \mu \times X\vec{f}$$

whence easily

$$X_{11}\phi = X\phi - \phi X.$$

By linearity, the same formula holds for any ϕ.

It follows that a matrix ϕ is an invariant (of type $(1,1)$) of X if, and only if, ϕ commutes with X. Therefore, if Y is a replica of X, Y commutes with every matrix which commutes with X. Making use of a well known theorem, this gives the following result:

THEOREM 1. *Any replica of a matrix X may be written in the form of a polynomial in X.*

2. Semi-simple and nilpotent components of a matrix. Using the same notations as in **1**, we shall say that a sub-space $\mathfrak{N}$ of $\mathfrak{M}$ is an $\{X\}$-module if $X\mathfrak{N} \subset \mathfrak{N}$. The $\{X\}$-module $\mathfrak{N}$ is said to be simple if it is $\neq \{0\}$ and if it does not contain any other $\{X\}$-modules than $\{0\}$ and itself. The space $\mathfrak{M}$ is said to be semi-simple (with respect to X) if $\mathfrak{M}$ is a sum of simple sub-modules. It is well known that, under this condition, $\mathfrak{M}$ can be represented as a direct sum of simple sub-modules.

If $\mathfrak{M}$ is semi-simple with respect to X, we shall also say that X is semi-simple over K. Well known examples show that X may be semi-simple over a certain field K and not be semi-simple over a larger field K' (this however only in the case where K'/K is not separable).

THEOREM 2. *A matrix X over a perfect field K may be represented in one and only one way in the form $X = Y + Z$, where Y is semi-simple over K, Z is nilpotent and Y, Z commute with X. Furthermore, Y and Z may be expressed as polynomials in X.*

This theorem is only a different form of the well known theorem of the reduction of a matrix to semi-diagonal form. We present here a new proof of it which is better adapted to our purposes.

Let us say that a polynomial $f(x)$ with coefficients in K is semi-simple (over K) if it is a product of *distinct* irreducible factors. We shall say that X satisfies a semi-simple equation in K if $f(X) = 0$ with some semi-simple f.

LEMMA 1. *If X satisfies a semi-simple equation, X is semi-simple.*

Assume that $f_1(X)f_2(X)\cdots f_r(X)=0$, with f_i irreducible $(1\leq i\leq r)$, f_i and f_j being relatively prime for $i\neq j$. Let $g_i=\prod_{j\neq i} f_j$; the h. c. d. of $g_1, g_2, \cdots, g_r$ is 1, and therefore we can find polynomials u_i such that

$$\Sigma_i u_i(x)g_i(x)=1. \tag{1}$$

We set $g_i(X)\mathfrak{M}=\mathfrak{M}_i$; $\mathfrak{M}_i$ is an $\{X\}$ module, and it follows from (1) that $\mathfrak{M}=\Sigma_i\mathfrak{M}_i$. We have $f_i(X)\mathfrak{M}_i=\{0\}$.

Let $\vec{e}$ be a vector $\neq 0$ in $\mathfrak{M}_i$ (if any), and assume that $f_i(x)=x^{m_i}+a_1x^{m_i-1}+\cdots+a_{m_i}$. The vectors $\vec{e}, X^h\vec{e}$ $(1\leq h\leq m_i)$ span a subspace $\mathfrak{N}$ of $\mathfrak{M}_i$ which is clearly an $\{X\}$-module. If $p(x)$ is a polynomial of degree $<m_i$, $p(x)$ is relatively prime to f_i: if $u(x)p(x)+v(x)f_i(x)=1$, we have $\vec{e}=u(X)p(X)\vec{e}$, whence $p(X)\vec{e}\neq 0$, which proves that $\vec{e}, Xe, \cdots, X^{m_i-1}\vec{e}$ are linearly independent and that $\dim\mathfrak{N}=m_i$. Furthermore $\mathfrak{N}$ is clearly the smallest $\{X\}$-module to contain $\vec{e}$. In the same way, we see that the smallest $\{X\}$-module to contain any vector $\vec{e'}\neq 0$ in $\mathfrak{N}$ is of dimension m_i, and therefore coincides with $\mathfrak{N}$. This means that $\mathfrak{N}$ is a simple $\{X\}$-module; therefore every element $\neq 0$ of $\mathfrak{M}_i$ belongs to a simple $\{X\}$-module, which proves Lemma 1.

Let now X be any matrix with coefficients in a given perfect field K. It satisfies at least one equation $f(X)=0$, where f is a polynomial with coefficients in K. We can find a semi-simple polynomial g which divides f and is such that f divides some power g^r of g (take for g the product of the distinct irreducible factors $f_1, f_2, \cdots, f_s$ of f).

We prove by induction on k the existence of a polynomial h_k such that $g(x-h_k(x)g(x))\equiv 0 \pmod{g^k}$. We may take $h_1=0$; since K is perfect, we have $dg/dx\neq 0$ and dg/dx is relatively prime to g (this follows at once from $g=f_1f_2\cdots f_s$, $df_j/dx\neq 0$). For any h we have

$$g(x-h(x)g(x))\equiv g(x)-(dg/dx)h(x)g(x) \pmod{g^2}.$$

Since dg/dx is relatively prime to g, the identity of Bezout furnishes us with a polynomial h_2 such that $1-(dg/dx)h_2\equiv 0 \pmod g$ whence $g(x-h_2g)\equiv 0 \pmod{g^2}$.

Suppose now that the existence of h_k has been proved for some $k\geq 2$. We set $g(x-h_k(x)g(x))=g^k(x)\phi(x)$, $y=x-h_2(x)g(x)$, whence $g(y-h_k(y)g(y))=g^k(y)\phi(y)\equiv 0 \pmod{g^{2k}(x)}$; on the other hand

$y - h_k(y)g(y) \equiv x \pmod{g(x)}$, and we can therefore write $y - h_k(y)g(y) = x - h_{k+1}(x)g(x)$; then $g(x - h_{k+1}(x)g(x))$ is divisible by $g^{2k}(x)$ and, *a fortiori*, by $g^{k+1}(x)$, which proves the existence of h_{k+1}.

We set now $h(x) = h_r(x)$, $Y = X - h(X)g(X)$, $Z = h(X)g(X)$. We have $X = Y + Z$, and Y, Z commute with X; since $g^r(X) = 0$, we have $g(Y) = 0$, and Y is semi-simple; for the same reason, we have $Z^r = 0$, and Z is nilpotent. The existence of a decomposition of the type prescribed in Theorem 2 is thereby proved. There remains to prove the statement of uniqueness.

Let us consider any decomposition $X = Y' + Z'$ with Y' semi-simple and Z' nilpotent, both commuting with X. We have $Y' - Y = Z - Z'$. Since Y', Z' commute with X, they also commute with Y and Z. Therefore, in particular, $Z - Z' = U$ is nilpotent as the difference of two commuting nilpotent matrices. By the Taylor expansion, we have

$$(2) \qquad g(Y') = U((dg/dx)(Y) + Ug_2(Y) + \cdots + U^{N-1}g_N(Y))$$

where $g_2, \cdots, g_N$ are polynomials. It follows that $g(Y')$ is nilpotent; $g^M(Y') = 0$ if $U^M = 0$. Since Y' is semi-simple, we have $\mathfrak{M} = \Sigma_i \mathfrak{M}'_i$ where each $\mathfrak{M}'_i$ is a simple $\{Y'\}$-module. Let f'_i be a polynomial of smallest possible degree such that $f'_i(Y')\mathfrak{M}' = \{0\}$. If $f'_i = pq$, p and q being polynomials, p of degree less than f'_i, the set of vectors $\vec{e} \in \mathfrak{M}'_i$ such that $p(Y')\vec{e} = 0$ is a $\{Y'\}$-module which is $\neq \mathfrak{M}'_i$; therefore this set reduces to $\{0\}$, and $p(Y')$ induces a regular linear endomorphism in $\mathfrak{M}'_i$ (i. e. an endomorphism which has an inverse). It follows immediately that $\phi(Y')\mathfrak{M}'_i = \{0\}$ whence the degree of $p = 0$: f'_i is irreducible. If now g' is the l. c. m. of the polynomials f'_i, we have $g'(Y') = 0$ and g' is semi-simple. Comparing the equations $g^M(Y') = 0$, $g'(Y') = 0$, we obtain easily $g(Y') = 0$.

Suppose now that $U^M = 0$, $M > 1$. Then (2) gives $U^{M-1}(dg/dx)(Y) = 0$. On the other hand, we have $h_2(x)(dg/dx) \equiv 1 \pmod{g(x)}$, whence $h_2(Y)(dg/dx)(Y) = E$ (the unit matrix) and $U^{M-1} = 0$. Therefore we have $U = 0$, $X = X'$, $Y = Y'$, which completes the proof of Theorem 2.

Remark. We have proved incidentally the converse of Lemma 1, i. e. that a semi-simple matrix satisfies a semi-simple equation.

If the field K is perfect, a polynomial which is semi-simple over K remains semi-simple over any larger field. Therefore we have the following result:

LEMMA 2. *A matrix which is semi-simple over a perfect field remains such over any larger field.*

Remark. The assertion of existence in Theorem 2 could be deduced easily

from the theorem that any algebra over a perfect field is the direct sum of the radical and of a semi-simple algebra.

We shall say that the matrices Y, Z whose existence and uniqueness are asserted in Theorem 2 are the *semi-simple and nilpotent components* of X.

Using the notations of **1**, we have

$$X_{r,s} = Y_{r,s} + Z_{r,s}. \tag{3}$$

Observe now that, if Z, Z' are nilpotent matrices, the matrix $Z \dotplus Z' = Z \times E' + E \times Z'$ is nilpotent as the sum of two nilpotent matrices which commute with each other (E, E' are unit matrices of suitable degrees). It follows immediately that $Z_{r,s}$ is nilpotent. We shall prove that $Y_{r,s}$ is semi-simple.

Since Y is semi-simple over a perfect field K, it remains semi-simple over the algebraic closure $\bar{K}$ of K. Hence we may write $Y = \Gamma D \Gamma^{-1}$, where D is a diagonal matrix and Γ a regular matrix, the coefficients of D and Γ belonging to $\bar{K}$. (This follows immediately from the fact that a simple $\{Y\}$-module over $\bar{K}$ is necessarily of dimension 1). Therefore it is sufficient to prove that $D_{r,s}$ is semi-simple. But $D_{r,s}$ may be computed directly without any difficulty; it turns out that $D_{r,s}$ is again diagonal (i. e. semi-simple). Moreover, if $\alpha_1, \alpha_2, \cdots, \alpha_n$ are the elements in the main diagonal of D, the elements of the main diagonal of $D_{r,s}$ are all the quantities

$$-\Sigma_1^r \alpha_{i_\lambda} + \Sigma_1^s \alpha_{j_\mu} \qquad (1 \leq i_\lambda \leq n,\ 1 \leq j_\mu \leq n,\ 1 \leq \lambda \leq r,\ 1 \leq \mu \leq s). \tag{4}$$

Since the matrices $Y_{r,s}$, $Z_{r,s}$ in (3) obviously commute with $X_{r,s}$, they are the semi-simple and nilpotent components of $X_{r,s}$.

3. Some further results on replicas of matrices.

LEMMA 1. *If Y, Z are the semi-simple and nilpotent components of a matrix X over a perfect field, and if $\vec{e}$ is a vector such that $X\vec{e} = 0$, then $Y\vec{e} = 0$, $Z\vec{e} = 0$.*

We have $Y\vec{e} = -Z\vec{e}$; since Y, Z commute, it follows easily that $Y^k\vec{e} = (-1)^k Z^k \vec{e}$ for all k, whence $Y^M\vec{e} = 0$ for some integer M. But we have also $g(Y)\vec{e} = 0$ with a semi-simple polynomial g. If d is the h. c. d. of $g(y)$ and y^M, we have, by the Bezout identity, $d(Y)\vec{e} = 0$. But $d(y) = y$ since g is semi-simple, which proves Lemma 1.

Since $Y_{r,s}$ and $Z_{r,s}$ are the semi-simple and the nilpotent components of $X_{r,s}$ and $Z_{r,s}$ it follows immediately that we have

LEMMA 2. *If X is a matrix in a perfect field K, the semi-simple and the nilpotent components of X are replicas of X.*

We shall now determine the replicas of a semi-simple matrix Y over a perfect field K. A replica of Y is of the form $p(Y)$, p being some polynomial. Let $\alpha_1, \alpha_2, \cdots, \alpha_n$ be the characteristic roots of Y, so that Y may be transformed (in the algebraic closure $\bar{K}$ of K) into the diagonal matrix

$$D = \begin{pmatrix} \alpha_1 & & & 0 \\ & \alpha_2 & & \\ & & \ddots & \\ 0 & & & \alpha_n \end{pmatrix}.$$

The same transformation carries $p(Y)$ into the matrix

$$D' = \begin{pmatrix} p(\alpha_1) & & & 0 \\ & p(\alpha_2) & & \\ & & \ddots & \\ 0 & & & p(\alpha_n) \end{pmatrix}.$$

The matrix $D_{r,s}$ is a diagonal matrix whose diagonal elements are given by (4). Since every vector which is mapped on 0 under $D_{r,s}$ is also mapped on 0 under $D'_{r,s}$, we see that the condition $-\Sigma_1{}^r\alpha_{i\lambda} + \Sigma_1{}^s\alpha_{j\mu} = 0$ implies $-\Sigma_1{}^r p(\alpha_{i\lambda}) + \Sigma_1{}^s p(\alpha_{j\mu}) = 0$, and conversely this condition is sufficient to make $D'_{r,s}$ a replica of $D_{r,s}$.

Therefore we have the following result:

THEOREM 3. *Let Y be a semi-simple matrix over a perfect field K, and let $\alpha_1, \alpha_2, \cdots, \alpha_n$ be the characteristic roots of Y. If p is a polynomial with coefficients in K, the matrix $p(Y)$ is a replica of Y if and only if the following condition is realized: if $\Sigma_i k_i \alpha_i = 0$ is any linear relation between the α_i's with rational integral coefficients, then the corresponding relation $\Sigma_i k_i p(\alpha_i) = 0$ holds between the elements $p(\alpha_i)$.*

An easy consequence of Theorem 3 is

THEOREM 4. *Let K be a field of characteristic 0. A necessary and sufficient condition for a matrix X to be nilpotent is that $Sp.\ XX' = 0$ for every replica X' of X.*

If X is nilpotent, and if X' is a replica of X, X' commutes with X and therefore XX' is nilpotent, whence Sp. $XX' = 0$.

Assume that X is not nilpotent. It has a semi-simple component $Y \neq 0$. Let $\alpha_1, \alpha_2, \cdots, \alpha_n$ be the characteristic roots of Y; they generate a vector space P of finite dimension over the field R of rational numbers. Let $\{\rho_1, \rho_2, \cdots, \rho_r\}$ be a base of P: we have $\alpha_i = \sum_{j=1}^{r} a_{ij}\rho_i$ and one at least of the numbers a_{ij}, say a_{ij_0}, is $\neq 0$. We set $\beta_i = a_{ij_0}$; if $\Sigma_i k_i \alpha_i = 0$ is a linear relation with integral coefficients between $\alpha_1, \alpha_2, \cdots, \alpha_n$ we have $\Sigma_i k_i a_{ij} = 0$ $(1 \leq j \leq h)$, whence $\Sigma_i k_i \beta_i = 0$. We can find a polynomial $p(x)$ (whose coefficients are algebraic over K) such that $p(\alpha_i) = \beta_i$ $(1 \leq i \leq n)$. The matrix $X'' = p(X)$ is a replica of Y. Let K' be the field generated by the coefficients of X''; if $\{\omega_1, \cdots, \omega_s\}$ is a linear base of K' over K, we may write $X'' = \Sigma_k \omega_k X'_k$, where each X'_k is a matrix with coefficients in K. Each X'_k is clearly a replica of X''. On the other hand, we have Sp. $XX'' =$ Sp. YX'' + Sp. ZX''; since Z commutes with Y and X'' is a replica of Y, Z commutes with X'', and Sp. $ZX'' = 0$, whence Sp. $XX'' =$ Sp. $YX'' = \Sigma_i \alpha_i \beta_i = \Sigma_i a^2_{ij_0}\rho_{j_0} + \sum_{i,j \neq j_0} a_{ij}a_{ij_0}\rho_j \neq 0$ since $\Sigma_i a^2_{ij_0} \neq 0$. It follows that Sp. $XX'_k \neq 0$ for at least one k, which proves Theorem 4.

LEMMA 3. *If the matrix X' is a replica of the matrix X, $X'_{r,s}$ is a replica of $X_{r,s}$.*

Consider the space $(\mathfrak{M}_{r,s})_{r',s'}$. If $\mathfrak{N}$, $\mathfrak{P}$, $\mathfrak{Q}$ are vector-spaces, there exist natural isomorphisms between the spaces $\mathfrak{N} \times \mathfrak{P} \times \mathfrak{Q}$, $(\mathfrak{N} \times \mathfrak{P}) \times \mathfrak{Q}$, $\mathfrak{N} \times (\mathfrak{P} \times \mathfrak{Q})$ and also between the spaces $\mathfrak{N} \times \mathfrak{P}$, $\mathfrak{P} \times \mathfrak{N}$; there exists also a natural isomorphism between $(\mathfrak{N}^*)^*$ and $\mathfrak{N}$. It follows easily from these remarks that there exists a natural isomorphism between $(\mathfrak{M}_{r,s})_{r',s'}$ and $\mathfrak{M}_{rs'+sr',\, rr'+ss'}$; and that the endomorphisms $(X_{r,s})_{r',s'}$ and $X_{rs'+sr',\, rr'+ss'}$ correspond to each other under this isomorphism. Since every invariant of type $(rs' + sr', rr' + ss')$ of X is also an invariant of the same type of X', Lemma 3 is proved.

It follows by Theorem 1, that $X'_{r,s}$ may be written as a polynomial $P_{r,s}(X_{r,s})$ in $X_{r,s}$. We may furthermore assume that $P_{r,s}(0) = 0$. In fact, if $X_{r,s}$ is a regular matrix, it satisfies at least one equation $g(X_{r,s}) = 0$ with a g such that $g(0) \neq 0$, and therefore the unit matrix $E_{r,s}$ may be written as a polynomial $e_{r,s}(X_{r,s})$ with $e_{r,s}(0) = 0$, which proves our assertion in this case. If $X_{r,s}$ is not regular, there exists at least one vector $\vec{e} \neq 0$ in $\mathfrak{M}_{r,s}$ such that $X_{r,s}\vec{e} = 0$, whence $P_{r,s}(X_{r,s})\vec{e} = P_{r,s}(0)\vec{e} = 0$, $P_{r,s}(0) = 0$. Hence we have

LEMMA 4. *If X' is a replica of X, $X'_{r,s}$ may be written in the form $p_{r,s}(X_{r,s})$ where $p_{r,s}$ is a polynomial such that $p_{r,s}(0) = 0$.*

Remark. We see at once that the condition expressed in Lemma 4 is not only necessary but also sufficient for X' to be a replica of X.

THEOREM 5. *Let X, X' be matrices over a perfect field K; let Y, Y' be the semi-simple components of X, X' and Z, Z' be their nilpotent components. In order for X' to be a replica of X, it is necessary and sufficient that Y' be a replica of Y and Z' a replica of Z.*

1) The condition is sufficient. Assume that the condition is satisfied, and let $\vec{e}$ be an invariant of type (r, s) of X. We have seen above that it is also an invariant of Y and of Z (cf. Lemma 1, **3**); therefore $\vec{e}$ is an invariant of Y' and of Z', and also of X': X' is a replica of X.

2) The condition is necessary. Assume that X' is a replica of X. Let $\vec{e}$ be an invariant of Y, of type (r, s): we have $X_{r,s}\vec{e} = Z_{r,s}\vec{e}$. Since $X_{r,s}$ commutes with $Z_{r,s}$, we have also $X^k{}_{r,s}\vec{e} = Z^k{}_{r,s}\vec{e}$ $(k = 1, 2, \cdots)$, whence $X^M{}_{rs}\vec{e} = 0$ for some M. By induction on k, we prove that $X^{M-k}_{r,s}Y'^k_{r,s}\vec{e} = 0$ $(k = 0, 1, \cdots, M)$. It is true for $k = 0$; assume that it holds for some $k < M$; we have $X_{r,s}(X^{M-k-1}_{r,s}Y'^k_{r,s}e) = 0$, whence $Y'_{r,s}(X^{M-k-1}_{r,s}Y'^k_{r,s}e) = 0$ because Y' is a replica of X' which is a replica of X, so that Y' is a replica of X. It follows that $X^{M-k-1}_{r,s}Y'^{k+1}_{r,s}e = 0$ because $Y'_{r,s}$ commutes with $X_{r,s}$, which proves our assertion for $k + 1$. Hence we have $Y'^M_{r,s}\vec{e} = 0$. Using (as we have done already) the fact that Y' is semi-simple, it follows that $Y'_{r,s}\vec{e} = 0$, which shows that Y' is a replica of Y.

Consider now the sub-space $\mathfrak{N}$ of $\mathfrak{M}_{r,s}$ formed by the invariants of Z. Since $Z_{r,s}$ may be written as a polynomial in $X_{r,s}$ and $X'_{r,s}$ commutes with $X_{r,s}$, $X'_{r,s}$ commutes with $Z_{r,s}$ whence $X'_{r,s}\mathfrak{N} \subset \mathfrak{N}$. Let $\bar{X}_{r,s}$, $\bar{X}'_{r,s}$, $\bar{Y}_{r,s}$ denote the endomorphisms induced on $\mathfrak{N}$ by $X_{r,s}$, $X'_{r,s}$, $Y_{r,s}$: we have $\bar{X}_{r,s} = \bar{Y}_{r,s}$. But we know that $Y_{r,s}$ is semi-simple; hence the same holds for $\bar{Y}_{r,s}$ and $\bar{X}_{r,s}$. We have $\bar{X}'_{r,s} = p_{r,s}(\bar{X}_{r,s})$, where $p_{r,s}$ is a polynomial, and it is easy to see that every polynomial in a semi-simple matrix is itself a semi-simple matrix. Therefore $\bar{X}'_{r,s}$ is semi-simple.

Since $Y'_{r,s}$, $Z'_{r,s}$ are polynomials in $X'_{r,s}$, they map $\mathfrak{N}$ into itself and induce endomorphisms $\bar{Y}'_{r,s}$, $\bar{Z}'_{r,s}$ on $\mathfrak{N}$; $\bar{Y}'_{r,s}$ is semi-simple, $\bar{Z}'_{r,s}$ is nilpotent; we have $\bar{X}'_{r,s} = \bar{Y}'_{r,s} + \bar{Z}'_{r,s}$ and $\bar{Y}'_{r,s}$, $\bar{Z}'_{r,s}$ commute with $\bar{X}'_{r,s}$; by the

uniqueness of the decomposition in semi-simple and nilpotent components, we have $\bar{Z}'_{r,s} = 0$, $Z'_{r,s}\mathfrak{N} = \{0\}$, which proves that every invariant of Z is also an invariant of Z'.

Theorem 5 reduces the problem of finding all replicas of a matrix X to the same problem with the supplementary assumption that X is either semi-simple or nilpotent. The problem is solved in the semi-simple case by Theorem 3; there remains to consider the nilpotent case.

THEOREM 6. *If Z is a nilpotent matrix over a field K of characteristic 0, the only replicas of Z are the matrices tZ, $t \in K$.*

In order to prove Theorem 6, we shall make use of a particular case of a theorem due to Ado. The original proof being written in Russian, we give here a proof of the results which we need.

The elements of $\mathfrak{M}_{r,0} = \mathfrak{M}^* \times \mathfrak{M}^* \times \cdots \times \mathfrak{M}^*$ are the r-linear forms on $\mathfrak{M}$. Among these, we consider only those which are symmetric: they form a vector sub-space $\mathfrak{S}_r$ of $\mathfrak{M}_{r,0}$. Let $\{\lambda_1, \lambda_2, \cdots, \lambda_n\}$ be a base of $\mathfrak{M}^*$ over K. It is well known that $\mathfrak{S}_r$ has a natural isomorphism with the set Π_r of the homogeneous polynomials of degree r in $\lambda_1, \lambda_2, \cdots, \lambda_n$. The operation $Z_{r,0}$ may therefore be considered as an operation on Π_r. This being true for every r, Z defines an operation D_Z in the ring $K[\lambda_1, \lambda_2, \cdots, \lambda_n]$ of all polynomials in $\lambda_1, \lambda_2, \cdots, \lambda_n$. It is clear that D_Z is a linear operation. An easy computation shows that it is a *derivation*, i. e.

$$D_Z(PQ) = D_Z(P) \cdot Q + P \cdot D_Z Q$$

if $P, Q \in K[\lambda_1, \cdots, \lambda_n]$. As such, D_Z may be extended to a derivation of the field $\Sigma = K(\lambda_1, \cdots, \lambda_n)$ of the rational functions in $\lambda_1, \lambda_2, \cdots, \lambda_n$. An element $R \in \Sigma$ is called an *invariant* of Z if $D_Z R = 0$. The invariants of Z clearly form a subfield J of Σ.

Since Z is nilpotent, it may be assumed that the base $\{\lambda_1, \lambda_2, \cdots, \lambda_n\}$ has been selected in such a way that $D_Z\lambda_1 = 0$, and that $D_Z\lambda_{k+1}$ is a linear combination of $\lambda_1, \lambda_2, \cdots, \lambda_k$ $(1 \leq k < n)$. We shall prove that we can find elements $\lambda'_1, \lambda'_2, \cdots, \lambda'_n \in \Sigma$ with the following properties:

a) The field $K(\lambda'_1, \lambda'_2, \cdots, \lambda'_k)$ coincides with $\Sigma_k = K(\lambda_1, \lambda_2, \cdots, \lambda_k)$ $(1 \leq k \leq n)$;

b) $D_Z\lambda'_k$ is either equal to 0 or to 1.

c) If $\mathfrak{o}_k = K[\lambda'_1, \lambda'_2, \cdots, \lambda'_k]$, we have $\lambda_i \in \mathfrak{o}_k$ $(1 \leq i \leq k)$.

d) If $D_Z\mathfrak{o}_k$ is $\neq \{0\}$, it coincides with the whole ring $\mathfrak{o}_k$.

We set $\lambda'_1 = \lambda_1$: the conditions a), b), c), d) are satisfied for $k = 1$, as $D_Z\mathfrak{o}_1 = \{0\}$.

Assume that $\lambda'_1, \cdots, \lambda'_k$ have been constructed. We set $\mu_{k+1} = D_Z\lambda_{k+1}$; by c) we have $\mu_{k+1} \in \mathfrak{o}_k$. If $D_Z\mathfrak{o}_k = \{0\}$, $\mu_{k+1} = 0$, we may take $\lambda'_{k+1} = \lambda_{k+1}$. If $D_Z\mathfrak{o}_k = \{0\}$, $\mu_{k+1} \neq 0$, we take $\lambda'_{k+1} = \mu_{k+1}^{-1}\lambda_{k+1}$; we have $K(\lambda'_1, \cdots, \lambda'_{k+1}) = K(\lambda_1, \cdots, \lambda_k, \lambda'_{k+1}) = \Sigma_{k+1}$. Since $D_Z\mu_{k+1} = 0$, we have $D_Z\lambda'_{k+1} = 1$. We have $\lambda_{k+1} = \mu_{k+1}\lambda'_{k+1} \in \mathfrak{o}_k[\lambda'_{k+1}] = \mathfrak{o}_{k+1}$; finally let $P = \Sigma p_i\lambda_{k+1}^{\prime i}$ be an element of $\mathfrak{o}_{k+1}$, with $p_i \in \mathfrak{o}_k$: since $D_Z p_i = 0$, we have $P = D_Z(\Sigma(i+1)^{-1}p_i\lambda_{k+1}^{\prime i+1})$.

Assume now that $D_Z\mathfrak{o}_k \neq \{0\}$, whence $D_Z\mathfrak{o}_k = \mathfrak{o}_k$. There exists an element $\nu_{k+1} \in \mathfrak{o}_k$ such that $D_Z\nu_{k+1} = \mu_{k+1}$; we set in this case $\lambda'_{k+1} = \lambda_{k+1} - \nu_{k+1}$, whence $D_Z\lambda'_{k+1} = 0$, $K(\lambda'_1, \cdots, \lambda'_{k+1}) = K(\lambda_1, \cdots, \lambda_k, \lambda'_{k+1}) = \Sigma_{k+1}$, $\lambda_{k+1} \in \mathfrak{o}_k[\lambda'_{k+1}] = \mathfrak{o}_{k+1}$. Let $P = \Sigma_i p_i\lambda_{k+1}^{\prime i}$ be any element of $\mathfrak{o}_{k+1}$, with $p_i \in \mathfrak{o}_k$; since $D_Z\mathfrak{o}_k = \mathfrak{o}_k$ we have $p_i = D_Z q_i$, $q_i \in \mathfrak{o}_k$ and $P = D_Z Q$ with $Q = \Sigma_i q_i\lambda_{k+1}^{\prime i} \in \mathfrak{o}_{k+1}$.

Moreover, our induction gives also the following results: we have $D_Z\lambda'_k = 1$ for at most one k; each λ'_k may be written as a rational function of $\lambda_1, \lambda_2, \cdots, \lambda_n$ whose denominator is an invariant (in fact, a power of μ_{k+1}, if $D_Z\lambda'_{k+1} = 1$). Assuming that $Z \neq 0$, we have $D_Z\lambda'_k = 1$ for exactly one value k_0 of k; we set $u = \lambda'_{k_0}$, and we have $\Sigma = J(u)$. The field J is generated over K by the λ'_k, $k \neq k_0$, which shows that every element of J may be written as quotient of two polynomials in $\lambda_1, \lambda_2, \cdots, \lambda_n$ which are both invariants.

Let now Z' be a replica of Z. A polynomial in $\lambda_1, \lambda_2, \cdots, \lambda_n$ which is an invariant of Z is also an invariant of Z'; therefore we have $D_{Z'}J = \{0\}$. If we set $v = D_{Z'}u$, we have

$$D_{Z'}\phi = vD_Z\phi$$

for any $\phi \in \Sigma$. In particular, we have $D_{Z'}\lambda_i = vD_Z\lambda_i$ $(1 \leq i \leq n)$. Since $D_Z\lambda_i$ and $D_{Z'}\lambda_i$ are both linear forms in $\lambda_1, \lambda_2, \cdots, \lambda_n$, we see that if $D_Z\lambda_i$, $D_Z\lambda_j$ are linearly independent for at least one pair (i, j) of indices, we may conclude that $v \in K$, whence $-{}^tZ' = v(-{}^tZ)$, $Z' = vZ$. There remains to consider the case where any two of the forms $D_Z\lambda_i$ are linearly dependent. It may then be assumed without loss of generality that $D_Z\lambda_i = 0$ for $1 \leq i \leq n-1$, $D_Z\lambda_n = \lambda_{n-1}$, and Z^* maps $\mathfrak{M}^*$ onto the sub-space spanned by λ_{n-1}. Since $(Z')^*$ is a polynomial in Z^*, we have $D_{Z'}\lambda_i = 0$, $D_{Z'}\lambda_n = v\lambda_{n-1}$, $v \in K$ whence again $Z' = tZ$ which completes the proof of Theorem 6.

PRINCETON UNIVERSITY.

ON CONJUGATE TRIGONOMETRIC POLYNOMIALS.*

By G. SZEGÖ.

1. In a joint paper with A. C. Schaeffer [1] we discussed the following question: *Let D be a closed domain in the complex z-plane and z_0 a fixed point of D. Let us consider all polynomials $f(z)$ of given degree n for which $|\Re f(z)| \leqq 1$ in D and $f(z_0)$ is real. What is the largest possible value of $|\Im f(z)|$ if $f(z)$ is an arbitrary polynomial of the kind mentioned and z is arbitrary in D?* Under certain conditions on D and z_0 the inequality $|\Im f(z)| < A \log n$ was established where $A > 0$ depends only on D and z_0.

This theorem was well-known in the special case when D is a circle and z_0 is an interior point of D. However Mr. P. Erdös has called my attention to the fact that the best possible bound of $|\Im f(z)|$ is not known in this case.

In the present paper I wish to determine the best possible bound in case D is a circle and z_0 is the center of D. I prove the following

THEOREM. *Let $f(z)$ be a polynomial of degree n, $f(0)$ real, and*

$$|\Re f(z)| \leqq 1, \qquad |z| \leqq 1. \tag{1.1}$$

Then

$$|\Im f(z)| \leqq 2(n+1)^{-1} \sum_{1 \leqq m \leqq n,\ m \text{ odd}} \operatorname{cotg} \frac{m\pi}{2(n+1)} = M_n, \qquad |z| \leqq 1. \tag{1.2}$$

This constant M_n can not be replaced by a smaller one.

The asymptotic value of M_n as $n \to \infty$ is $(2/\pi) \log n$.

2. Another formulation of this theorem is as follows:

Let

$$u(\phi) = a_0 + a_1 \cos \phi + b_1 \sin \phi + \cdots + a_n \cos n\phi + b_n \sin n\phi \tag{2.1}$$

be a trigonometric polynomial of degree n with real coefficients satisfying the condition

$$|u(\phi)| \leqq 1, \qquad \phi \text{ real.} \tag{2.2}$$

Then the conjugate polynomial

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* Received March 9, 1942.

[1] A. C. Schaeffer and G. Szegö, "Polynomials whose real part is bounded on a given curve in the complex plane," *American Journal of Mathematics*, vol. 62 (1940), pp. 868-876.

(2.3) $v(\phi) = -b_1 \cos\phi + a_1 \sin\phi - \cdots - b_n \cos n\phi + a_n \sin n\phi$

satisfies the inequality

(2.4) $$|v(\phi)| \leqq M_n, \qquad \phi \text{ real},$$

where M_n has the same meaning as in (1.2). *This bound can not be replaced by a smaller one.*

Further discussion is devoted to the problem of determining all the cases in which $|v(\phi_0)| = M_n$ holds, ϕ_0 being a fixed real value. As we shall prove, the solution of this problem is different according as n is even or odd.

For n even $|v(\phi_0)| = M_n$ holds if, and only if, $u(\phi) = \pm u_0(\phi - \phi_0)$ where

(2.5) $$u_0(\phi) = \lambda_0(\phi) + \gamma \frac{1 + \cos(n+1)\phi}{1 + \cos\phi};$$

here $\lambda_0(\phi)$ is the special sine polynomial (6.1) and γ is real, $|\gamma| \leqq (n+1)^{-2}$.

For n odd $|v(\phi_0)| = M_n$ holds if and only if $u(\phi) = \pm \lambda_0(\phi - \phi_0)$.

3. It is sufficient to prove (2.4) for $\phi = 0$. Furthermore the sine polynomial

(3.1) $\lambda(\phi) = \frac{1}{2}(u(\phi) - u(-\phi)) = b_1 \sin\phi + b_2 \sin 2\phi + \cdots + b_n \sin n\phi$

satisfies the inequality $|\lambda(\phi)| \leq 1$ and its conjugate is

(3.2) $\mu(\phi) = \frac{1}{2}(v(\phi) + v(-\phi)) = -b_1 \cos\phi - b_2 \cos 2\phi - \cdots - b_n \cos n\phi$

so that $\mu(0) = v(0)$. Thus our theorem is equivalent to the following:

Let the sine polynomial

(3.3) $$\lambda(\phi) = b_1 \sin\phi + b_2 \sin 2\phi + \cdots + b_n \sin n\phi$$

satisfy the condition $|\lambda(\phi)| \leqq 1$, ϕ real. Then (with the same M_n as before)

(3.4) $$\left|\sum_{\nu=1}^{n} b_\nu\right| \leqq M_n.$$

This bound can not be replaced by a smaller one.

4. Next we prove the following

LEMMA. *By using the notation* (3.3) *we have the identity*

(4.1) $$\sum_{\nu=1}^{n} b_\nu = 2(n+1)^{-1} \sum_{1 \leqq m \leqq n,\ m \text{ odd}} \lambda\left(\frac{m\pi}{n+1}\right) \operatorname{cotg} \frac{m\pi}{2(n+1)}.$$

Indeed, for any trigonometric polynomial $U(\phi)$ of degree $2n+1$ we have as well-known

$$(4.2) \qquad \pi^{-1}\int_{-\pi}^{+\pi} U(\phi)\,d\phi = (n+1)^{-1}\sum_{m=-n}^{n+1} U\left(\frac{m\pi}{n+1}\right).$$

Apply this to the cosine polynomial [2]

$$(4.3) \quad U(\phi) = \lambda(\phi)\sum_{\nu=1}^{n}\sin\nu\phi = \lambda(\phi)\,\frac{\sin(n/2)\phi\,\sin[(n+1)/2]\phi}{\sin(\phi/2)}.$$

Then $[U(0) = U(\pi) = 0,\ U(-\phi) = U(\phi)]$

$$(4.4) \qquad \sum_{\nu=1}^{n} b_\nu = \pi^{-1}\int_{-\pi}^{+\pi}\lambda(\phi)\Big(\sum_{\nu=1}^{n}\sin\nu\phi\Big)\,d\phi$$

$$= 2(n+1)^{-1}\sum_{m=1}^{n}\lambda\left(\frac{m\pi}{n+1}\right)\frac{\sin\left(\frac{n}{2}\,\frac{m\pi}{n+1}\right)\sin\frac{m\pi}{2}}{\sin\left(\frac{1}{2}\,\frac{m\pi}{n+1}\right)}.$$

This can easily be identified with (4.1).

5. The inequality $\left|\sum_{\nu=1}^{n} b_\nu\right| \leqq M_n$ follows immediately from the Lemma. The only remaining question is whether the equality sign holds for a suitable sine polynomial. This is the case if and only if

$$(5.1) \qquad \lambda\left(\frac{m\pi}{n+1}\right) = +1 \ (\text{or } -1), \qquad 1 \leqq m \leqq n,\ m \text{ odd}.$$

These equations in combination with $|\lambda(\phi)| \leqq 1$ furnish

$$(5.2) \qquad \lambda'\left(\frac{m\pi}{n+1}\right) = 0, \qquad 1 \leqq m \leqq n,\ m \text{ odd}.$$

The total number of these conditions is n or $n+1$ according as n is even or odd. They determine the sine polynomial $\lambda(\phi)$ uniquely in the first case whereas in the second case the existence of such a polynomial has to be shown. Finally in both cases the inequality $|\lambda(\phi)| \leqq 1$ must be verified.

6. The special sine polynomial $\lambda_0(\phi)$ satisfying conditions (5.1) and (5.2) has the character of the so called step-parabolas. It can be obtained by the Hermite interpolation: [3]

$$(6.1) \quad \lambda_0(\phi) = 2(n+1)^{-2}\sin\phi \sum_{1\leqq m\leqq n,\ m \text{ odd}} \frac{1+\cos(n+1)\phi}{\left(\cos\phi - \cos\frac{m\pi}{n+1}\right)^2}\sin\frac{m\pi}{n+1}.$$

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[2] See, for instance, G. Pólya and G. Szegö, *Aufgaben und Lehrsätze aus der Analysis*, vol. 1 (Berlin, 1925). p. 77, p. 269, problem 16.

[3] Cf. for instance, G. Szegö, *Orthogonal Polynomials*, Colloquium Publications, vol. 23 (New York, 1939), p. 324, (14.1.9).

This is indeed a sine polynomial of degree n. The verification of the conditions mentioned does not cause any trouble. We only have to show that the polynomial

$$2(n+1)^{-2}\sin\phi\,\frac{1+\cos(n+1)\phi}{(\cos\phi-\cos m\pi/(n+1))^2}\sin m\pi/(n+1)$$

$$(6.2)\qquad =2(n+1)^{-2}(1-x_m^2)^{\frac{1}{2}}(1-x^2)^{\frac{1}{2}}\frac{1+T_{n+1}(x)}{(x-x_m)^2}=t(x),$$

$$x=\cos\phi,\ x_m=\cos m\pi/(n+1),\qquad T_{n+1}(x)=\cos(n+1)\phi,$$

satisfies the conditions $t(x_m)=1$, $t'(x_m)=0$. But from the well-known differential equation

$$(6.3)\qquad (1-x^2)T''_{n+1}(x)-xT'_{n+1}(x)+(n+1)^2T_{n+1}(x)=0$$

we conclude that (m odd)

$$(6.4)\quad T''_{n+1}(x_m)=(n+1)^2(1-x_m^2)^{-1},\ T'''_{n+1}(x_m)=3(n+1)^2x_m(1-x_m^2)^{-2}.$$

Taking into account the relation

$$(6.5)\quad 1+T_{n+1}(x)=\tfrac{1}{2}(x-x_m)^2T''_{n+1}(x_m)+\tfrac{1}{6}(x-x_m)^3T'''_{n+1}(x_m)+\cdots$$

we obtain

$$\begin{aligned}(6.6)\quad t(x)&=2(n+1)^{-2}(1-x_m^2)^{\frac{1}{2}}(1-x^2)^{\frac{1}{2}}\{\tfrac{1}{2}(n+1)^2(1-x_m^2)^{-1}\\&\quad+\tfrac{1}{2}(n+1)^2x_m(1-x_m^2)^{-2}(x-x_m)+\cdots\}\\&=\{1-x_m(1-x_m^2)^{-1}(x-x_m)+\cdots\}\\&\quad\cdot\{1+x_m(1-x_m^2)^{-1}(x-x_m)+\cdots\}\end{aligned}$$

from which the assertion follows.

7. Further we need to show that $|\lambda_0(\phi)|\leqq 1$ holds, ϕ real. Condition (5.2) furnishes $[\frac{1}{2}(n+1)]$ roots of $\lambda'_0(\phi)$ in $0<\phi<\pi$, condition (5.1) in combination with Rolle's theorem $[\frac{1}{2}(n+1)]-1$ further roots of $\lambda'_0(\phi)$ in the same interval. The total number is $2[\frac{1}{2}(n+1)]-1=n-1$ or n according as n is even or odd. But in the first case π is also a (double) root of $\lambda'_0(\phi)$ so that in both cases we find the maximum number of roots for the cosine polynomial $\lambda'_0(\phi)$ in $0<\phi<\pi$. Thus all these roots of $\lambda'_0(\phi)$ are simple and these are the only roots of $\lambda'_0(\phi)$ in $0<\phi<\pi$. Hence $\lambda_0(\phi)$ is monotonic in the intervals determined by those roots. Taking into account $\lambda_0(0)=0$ and the conditions (5.1) we find that $\lambda_0(\phi)\leqq 1$ in $0\leqq\phi\leqq\pi$. From (6.1) we see that in the same interval $\lambda_0(\phi)\geqq 0$.

This establishes the proof of our Theorem.

8. Finally we wish to determine all the cases in which $|v(0)| = M_n$ holds. The previous argument shows that in this case $u(\phi) = \pm u_0(\phi)$ where

$$u_0(\phi) = \lambda_0(\phi) + c_0(\phi); \tag{8.1}$$

here $c_0(\phi)$ is a certain cosine polynomial with real coefficients. Because of (2.2), (5.1) and (5.2) we must have

$$c_0\left(\frac{m\pi}{n+1}\right) = c'_0\left(\frac{m\pi}{n+1}\right) = 0, \qquad 1 \leqq m \leqq n,\ m \text{ odd}. \tag{8.2}$$

The number of these conditions is n or $n+1$ according as n is even or odd. Hence in the second case $c_0(\phi) \equiv 0$.

Let n be even. Then

$$c_0(\phi) = \gamma \frac{1+\cos(n+1)\phi}{1+\cos\phi}, \qquad \gamma \text{ a real constant}, \tag{8.3}$$

are the only polynomials satisfying (8.2). We have $|u_0(\pi)| = |c_0(\pi)| = |\gamma|\,(n+1)^2 \leqq 1$. Now we show that

$$0 \leqq h(\phi) = \lambda_0(\phi) + (n+1)^{-2}\frac{1+\cos(n+1)\phi}{1+\cos\phi} \leqq 1, \quad 0 \leqq \phi \leqq \pi. \tag{8.4}$$

Indeed,

$$\begin{aligned} &h[m\pi/(n+1)] = -h\{-[m\pi/(n+1)]\} = 1, \\ &h'[m\pi/(n+1)] = h'\{-[m\pi/(n+1)]\} = 0, \quad 1 \leqq m \leqq n,\ m \text{ odd}. \end{aligned} \tag{8.5}$$

Also $h(\pi) = 1$, $h'(\pi) = 0$. Furthermore Rolle's Theorem furnishes $n/2$ roots of $h'(\phi)$ in $0 < \phi < \pi$ and $n/2 - 1$ roots in $-\pi < \phi < 0$. Thus we find altogether

$$n/2 + 1 + n/2 + n/2 + n/2 - 1 = 2n$$

(mod 2π) distinct roots for $h'(\phi)$. Hence all these roots of $h'(\phi)$ are simple and these are the only roots of $h'(\phi)$. Consequently $h(\phi)$ is monotonic in the intervals defined by those roots. Since $h(0) = (n+1)^{-2}$ an easy discussion furnishes (8.4).

STANFORD UNIVERSITY,
CALIFORNIA.

RADICAL IDEALS.*

By REINHOLD BAER.

The radical has been introduced into the theory of rings with the intent of constructing a two-sided nilideal modulo which there do not exist nilpotent right-ideals different from 0. Thus it seems justified to define as a radical ideal every ideal meeting these requirements. Every ring possesses at least one radical ideal; and both the cross-cut and the sum of all the radical ideals are themselves radical ideals which may be called the lower and the upper radical respectively. It is possible that the upper and lower radicals are different and that neither of them is nilpotent. If all the radical ideals are equal, then we say that the radical exists; and this happens e. g., if every right-ideal, not 0, in the quotient ring modulo the lower radical contains a smallest right-ideal different from 0. If this latter condition is satisfied by every quotient ring of the ring under consideration, then a finite or transfinite power of the radical is 0.

Several applications of the theory of radical ideals are given: we prove a criterion for the existence of the identity element in rings that need not satisfy the minimum condition for right- (or left-) ideals; and we deduce the double chain condition for right-ideals from properties considerably weaker than the minimum condition.

It should be noted that we have restricted our attention throughout to the consideration of right-ideals.

1. Existence of the upper and lower radicals. The element x in the ring R is a *nilelement*, if $x^i = 0$ for some exponent i; and the right-ideal J in R may be termed a *nilideal*, if every element in J is a nilelement. The right-ideal J is *nilpotent*, if $J^i = 0$ for some exponent i. It is clear that nilpotent right-ideals are nilideals, though the converse need not be true.[1] It is well known [2] that the sum of a finite number of nilpotent right-ideals is a nilpotent right-ideal; and it is obvious that xJ is a nilpotent right-ideal, whenever J is a nilpotent right-ideal and x an element in R. From these two facts one readily deduces the following well known statement.

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[1] For an example cf. Köthe (1), p. 165 (the numbers in parentheses refer to the bibliography at the end of the paper).

[2] Cf. e. g. v. d. Waerden (1), p. 154.

LEMMA 1.1. *The sum $N = N(R)$ of all the nilpotent right-ideals in R is a two-sided nilideal in R.*

Note that the ideal $N(R)$ need not be nilpotent, as follows from an example due to Köthe.[3]

The ideal P in the ring R shall be termed *a radical ideal*,[4] if

(1. a) P is a two-sided ideal;

(1. b) P is a nilideal;

(1. c) the quotient-ring R/P does not contain nilpotent right-ideals different from 0.

There exist ideals in R which meet the requirements (1. a) and (1. b), e. g. the null-ideal. Thus we may form the sum [5] $U = U(R)$ of all the ideals P in R satisfying conditions (1. a) and (1. b). This ideal U is clearly a two-sided ideal in R and shall be called *the upper radical of R.*

There exist ideals in R which meet the requirements (1. a) and (1. c), e. g. the ideal $P = R$. Thus we may form the cross-cut $L = L(R)$ of all the ideals P in R which satisfy conditions (1. a) and (1. c). This ideal L is obviously a two-sided ideal in R and shall be termed *the lower radical of R.*

It is an immediate consequence of our definitions that every radical ideal is situated between the upper and the lower radical. But the main justification for our terminology may be seen in the following fact.

THEOREM 1. 2. *The upper and the lower radical of the ring R are radical ideals in R.*

Proof. It has been shown by Köthe [6] that the sum of all the two-sided nilideals in R is a nilideal. Denote by W the uniquely determined two-sided ideal in R which satisfies: $U \leq W$ and $W/U = N(R/U)$. There exists, by Lemma 1. 1, to every element x in W a positive integer i such that x^i is an element in the nilideal U; and hence every element in W is a nilelement. Now it follows from the definition of the upper radical that $W = U$, $N(R/U) = 0$; and thus we have shown that the upper radical of R is a radical ideal in R.

$L \leq U$, since we have just shown that the upper radical U is a radical ideal, and since the lower radical L is certainly part of every radical ideal in R. Consequently L is a nilideal, since U is a nilideal. Every nilpotent right-

[3] Cf. Köthe (1), p. 165.

[4] The existence of radical ideals will be assured by Theorem 1. 2 below.

[5] This ideal has been considered by Fitting (1), p. 21.

[6] Köthe (1), p. 170.

ideal in R/L has the form S/L for S a suitable right-ideal between L and R; and from the nilpotence of S/L we infer the existence of a positive integer t such that $S^t \leq L$. Suppose now that condition (1. c) is satisfied by the two-sided ideal T in R. Then $L \leq T$ and $((T+S)/T)^t = 0$, since $(T+S)^t \leq T + S^t \leq T + L = T$; and it follows from (1. c) that $(T+S)/T = 0$ or $T + S = T$ or $S \leq T$. Thus we have shown that S is part of every ideal T, satisfying (1. a) and (1. c); and hence $S \leq L$, proving[7] that L is a radical ideal in R.

Remarks. 1. The following construction of the lower radical may be of some interest (in particular in case the transfinite induction involved in it happens to stop after a finite number of steps):

(i) $Q(0) = 0$.

(ii) Suppose that the two-sided ideal $Q(u)$ has been defined for every $u < v$.

Case 1. If $v = w + 1$ is not a limit-ordinal, then $Q(v)$ is the uniquely determined ideal in R which contains $Q(w)$ and which satisfies: $Q(v)/Q(w) = N(R/Q(w))$; that $Q(v)$ is a two-sided ideal in R, is an immediate consequence of Lemma 1. 1.

Case 2. If v is a limit-ordinal, then denote by $Q(v)$ the join of all the ideals $Q(u)$ for $u < v$. It is readily verified that $Q(v)$ is a two-sided ideal in R.

(iii) Since the $Q(v)$ form an ascending chain of ideals in R, there exists a (smallest) ordinal z such that $Q(z) = Q(z+1)$; and we put $Q(z) = Q$.

It is clear from our construction of Q that Q is a two-sided ideal in R and that R/Q does not contain nilpotent right-ideals different from 0. Hence it follows from the definition of the lower radical L of R that $L \leq Q$.

Suppose next that the ideal T in R meets the requirements (1. a) and (1. c). It is clear that $Q(0) \leq T$; and thus we may assume that every $Q(u)$ for $u < v$ is part of T. Then it is readily verified that $Q(v)$ is also part of T; and thus it follows by complete induction that Q is part of T. But this shows that $Q \leq L$; and thus we have shown that the ideal Q just constructed is the lower radical L of R.

2.[8] If T is a two-sided ideal between L and U, then the upper radical of R/T is U/T; and if T is a radical ideal, then the lower radical of R/T is 0.

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[7] The author is indebted to the referee for this proof, which is much simpler than the author's original proof, of the fact that L is a radical ideal.

[8] This remark is due to the referee.

3. It has been pointed out before that every radical ideal is situated between the upper and lower radical; and it is a consequence of Theorem 1.2 that every subideal of the upper radical is a nilideal. It may happen that there exist two-sided ideals between the upper and the lower radical which are not radical ideals; an example for this phenomenon will be constructed in section **2**.

4. It is a consequence of Theorem 1.2 that *the upper and lower radicals are equal, if U/L is a nilpotent ideal in R/L.* In general, however, the upper and lower radicals may be different, as will be seen from the example constructed in section **2**.

If the upper and the lower radical are equal, then this ideal may be termed *the radical $K = K(R)$ of the ring R*; and we say then that *the radical of R exists.* It should be noted that the radical need not be nilpotent, as may be seen from an example due to Köthe.[9]

2. Existence of different radical ideals. In this section we construct a ring with the following properties:

(i) Every element in the ring is a nilelement so that the ring is its own upper radical U.

(ii) The ring does not contain nilpotent right-ideals different from 0 so that its lower radical is 0.

(iii) The ring contains a two-sided ideal which is not a radical ideal.

Denote by G an abelian group which is the direct sum of a countably infinite number of infinite cyclic groups; and denote by $b(0), b(1), b(-1), \cdots, b(i), b(-i), \cdots$ a basis of G. Then there exists one and only one endomorphism (= homomorphism) $u(i)$, for $i = 1, 2, \cdots$, of G such that

$$b(j)u(i) = \begin{cases} 0, & \text{if } j \equiv 0 \bmod 2^i \\ b(j-1), & \text{if } j \not\equiv 0 \bmod 2^i. \end{cases}$$

The following statement may be easily verified by complete induction:

$$(*)\quad b(j)u(i_0)\cdots u(i_m) = \begin{cases} b(j-m-1), \text{ if } j \not\equiv n \bmod 2^{i_n} \text{ for } 0 \leq n \leq m \\ 0, \text{ if } j \equiv n \bmod 2^{i_n} \text{ for at least one } n \text{ with } 0 \leq n \leq m. \end{cases}$$

Denote by U the ring of endomorphisms of G which is generated by the endomorphisms $u(1), u(2), \cdots$.

Every element $x \neq 0$ in U has the form: $x = \sum_{i=1}^{k} x_i$ where $x_i = \pm u((i,0))$

[9] Köthe (1), p. 165.

$\cdots u(_{(i,}, m_i))$, $0 < (i, j)$, $0 \leq m_i$. If h is the maximum of all the numbers $2^{(i,n)}$, then it is an immediate consequence of (*) that $b(j)x_{i_1} \cdots x_{i_h} = 0$ for every j, since the product of endomorphisms contains at least h factors $u(s)$ with $2^s \leq h$. Consequently this product is 0 and this implies that $x^h = 0$. The ring U has therefore property (i).

For a proof of property (ii) we need a closer analysis of the structure of the endomorphisms in U. If x is an endomorphism different from 0 in U, then $x = \sum_{i=0}^{m} y(i)$, $y(i) = \sum_{j=1}^{k(i)} e(i,j)y(i,j)$, $e(i,j) = \pm 1$, $y(i,j) = u((i,j;0))$ $\cdots u((i,j;i))$, $0 < (i,j;n)$, $0 \leq k(i)$, and $y(n) \neq 0$. Denote by r some integer satisfying: $m + 1 < 2^r$ and $(i, j; n) \leq r$ for $i = 0, \cdots, m$; $j = 1, \cdots, k(i)$; $n = 0, \cdots, i$.

If h is some preassigned positive integer, then denote by s an integer satisfying $2^{r+1}h \leq 2^s$. We proceed to prove that $(xu(s)^{2^r-m-1})^h \neq 0$; a fact that implies the impossibility of nilpotence of right-ideals different from 0.

It will be convenient to put $z(i,j) = y(i,j)u(s)^{2^r-m-1}$.

Since $y(m) \neq 0$, there exists an integer t such that $b(t)y(m) \neq 0$. Hence we may assume that the positive integer k has been determined in such a way that

$$b(t)y(m,j) = \begin{cases} b(t-m-1) & \text{for } j = 1, \cdots, k \\ 0 & \text{for } k < j. \end{cases}$$

Then $b(t)y(m) = \sum_{j=1}^{k} e(m,j)b(t-m-1) \neq 0$; and this shows that $\sum_{j=1}^{k} e(m,j) \neq 0$, a fact we shall have to use later on.

If $0 \leq i_v \leq m$, $1 \leq j_v \leq k(i_v)$ for $1 \leq v \leq h$, and if e is 0 or 1, then it follows from (*) that $b(t + eh2^r)z(i_1, j_1) \cdots z(i_h, j_h)$ is either 0 or $b(t + eh2^r - \sum_{v=1}^{h} (i_v + 1 + 2^r - m - 1)) = b(t + (e-1)h2^r + mh - \sum_{v=1}^{h} i_v)$ and this latter element is equal to $b(t + (e-1)h2^r)$ if, and only if, $i_1 = \cdots = i_h = m$.

If $k < j_v$ for some v, then $b(t)y(m, j_v) = 0$; and thus there follows from (*) the existence of an integer n such that $0 \leq n \leq m$ and $t \equiv n \bmod 2^{(m, j_v; n)}$. But then we find that $t + eh2^r \equiv t \equiv n \equiv n + (v-1)2^r \bmod 2^{(m, j_v; n)}$ and hence we may deduce from (*) that $b(t + eh2^r)z(m, j_1) \cdots z(m, j_h) = 0$. The results obtained in the last two paragraphs may be stated as follows:

If $b(t + eh2^r)z(i_1, j_1) \cdots z(i_h, j_h) = b(t + (e-1)h2^r)$, then $i_1 = \cdots = i_h = m$ and $1 \leq j_v \leq k$.

Assume now the existence of integers j_v, j'_v such that $1 \leq j_v, j'_v \leq k$ and

$b(t)z(m,j_1)\cdot\cdot\cdot z(m,j_h)=0$, $b(t+h2^r)z(m,j'_1)\cdot\cdot\cdot z(m,j'_h)=0$. Then we infer from (*) the existence of integers n, v satisfying: $1\leq v\leq h$ and

$$t\equiv\begin{cases} n+(v-1)2^r \bmod 2^{(m,j_v;n)} & \text{for } 0\leq n\leq m \\ n+(v-1)2^r \bmod 2^s & \text{for } m<n<2^r. \end{cases}$$

If $0\leq n\leq m$, then $t\equiv n \bmod 2^{(m,j_v;n)}$; and we could deduce from (*) that $b(t)y(m,j_v)=0$ which is impossible, since $j_v\leq k$. Thus there exist integers n, v such that $m<n<2^r$, $1\leq v\leq h$ and $t\equiv n+(v-1)2^r \bmod 2^s$; and likewise there exist integers n', v' such that $m<n'<2^r$, $1\leq v'\leq h$ and $t+h2^r\equiv n'+(v'-1)2^r \bmod 2^s$. Since $r<s$, it follows from these two congruences that $n\equiv n' \bmod 2^r$; and since $m<n$, $n'<2^r$, we deduce that $n=n'$. Sonsequently $(v-1+h)2^r\equiv(v'-1)2^r \bmod 2^s$ or $h+v-v'\equiv 0 \bmod 2^{s-r}$. Since $1\leq v,\ v'\leq h$, we have $0<h+v-v'<2h\leq 2^{s-r}$, a contradiction. Thus we have shown that there exists an integer t' such that

$$b(t')z(i_1,j_1)\cdot\cdot\cdot z(i_h,j_h)=b(t'-h2^r)$$

if, and only if, $i_1=\cdot\cdot\cdot=i_h=m$ and $1\leq j_v\leq k$ for $v=1,\cdot\cdot\cdot,h$.

Now it is readily verified that

$$b(t')(xu(s)^{2^r-m-1})^h=(\sum_{j=1}^{k}e(m,j))^h b(t'-h2^r)+\sum_{i\neq t'-h2^r}c(i)b(i)$$
$$\neq 0,$$

since the factor of $b(t'-h2^r)$ is, by a previous remark, different from 0.

This completes the proof of the fact that (ii) is satisfied by the ring U.

To prove (iii) let us consider the two-sided ideal T in U which is generated by $u(2)$, $u(3),\cdot\cdot\cdot$. Every element in T is a linear combination of products $u(i_0)\cdot\cdot\cdot u(i_m)$ with the restriction that none of these products is a power of $u(1)$.

We note that $b(j)u(1)=\begin{cases} 0 \text{ for even } j \\ b(j-1) \text{ for odd } j \end{cases}$. Hence $u(1)^2=0$; and it is readily verified that $U^2\leq T$. Thus all we have to show is that $u(1)$ is not an element in T.

Every element $x\neq 0$ in T has the form: $x=\sum_{i=2}^{k}c(i)u(i)+x'$ for x' in U^2 so that $b(t)x'=\sum_{j\neq t-1}d(j)b(j)$. If $x=u(1)$, then it follows from $b(1)u(i)=b(0)$ that $\sum_{i=2}^{k}c(i)=1$; and it follows from $b(2)u(i)=\begin{cases} 0 \text{ for } i=1 \\ b(1) \text{ for } i\neq 1 \end{cases}$ that $\sum_{i=2}^{k}c(i)=0$. This contradiction shows that $u(1)$ is not in T so that $U^2\leq T<U$, as was required.

If we adjoin to the ring U of endomorphisms of the abelian group G the identity-endomorphism, we obtain a ring R whose upper radical is U, whose lower radical is 0 and which contains the two-sided nilideal T such that R/T contains nilpotent right-ideals different from 0.

3. Characterization of the upper radical. The right-ideal J in the ring R is termed *a minimal right-ideal*, if $0 < J$, and if there does not exist a right-ideal J' such that $0 < J' < J$.

THEOREM 3.1. *If T is a radical ideal in the ring R, and if every right-ideal different from 0 in the quotient ring R/T contains a minimal right-ideal, then T is the upper radical U of R and the upper radical of R contains every nilideal*[10] *in R.*

Proof. If the nilideal J in R is not part of T, then there exists an ideal V between T and $T+J$ such that V/T is a minimal right-ideal in R/T. Since V/T is part of $(T+J)/T$, and since J is a nilideal, V/T is a nilideal. A minimal right-ideal is either nilpotent or idempotent; and since T is a radical ideal in R, it follows that V/T is idempotent. But it is well known[11] that idempotent minimal right-ideals contain idempotent elements not 0. Thus the nilideal V/T contains an idempotent element not 0, a contradiction. Hence it follows that T contains every nilideal in R. Since T is a radical ideal, it is a two-sided nilideal and therefore part of the upper radical. Since the upper radical is the sum of all the two-sided nilideals, it follows from what we have shown just now that U is part of T, i. e. $U = T$ contains every nilideal.

The right-ideal J is termed *a maximal right-ideal in the ring R*, if $J < R$, and if there does not exist a right-ideal J' such that $J < J' < R$. The element e in R is a *left-identity element*, if $ex = x$ for every element x in the ring R.

LEMMA 3.2. *If the ring R contains a left-identity element e, then each nilideal is part of every maximal right-ideal.*

Proof. If the nilideal V were not part of the maximal right-ideal J, then $R = V + J$ and there would exist elements v and j in V and J respectively such that $e = v + j$. By complete induction we may show the existence of elements $j(i)$ in J such that $e = v^i + j(i)$, since

$$e = e^2 = (j+v)v^i + ej(i) = v^{i+1} + jv^i + j(i) = v^{i+1} + j(i+1).$$

[10] This shows that under the hypotheses of Theorem 3.1 the upper radical meets all the requirements, concerning right-ideals, imposed upon the radical by Köthe (1), p. 169.

[11] Cf. e.g. v. d. Waerden (1), p. 157, Hilfssatz 3.

But v is a nilelement; and hence it follows that e is in J and that therefore $J = R$, an impossibility which proves our contention.

THEOREM 3.3.[12] *If the ring R contains a left-identity element, and if every right-ideal different from 0 in the quotient ring R/U contains a minimal right-ideal, then the upper radical U of R is the cross-cut of all the maximal right-ideals in R.*

Remark. The example in section **2** shows the impossibility of omitting the hypothesis concerning R/U; and Theorem 9.6 below shows the need for assuming the existence of a left-identity element in R.

Proof. If the intersection J of all the maximal right-ideals in R/U were different from 0, then J would contain a minimal right-ideal J'; and consequently we could infer[13] the existence of an idempotent $e \neq 0$ in J'. It is well known that R/U is the direct sum of the right-ideals J' and Z where J' consists of all the elements $v = ev$ and where Z consists of the elements z satisfying $ez = 0$. Thus the crosscut of Z and J' is 0. Since J' is a minimal right-ideal in R/U, Z is a maximal right-ideal in R/U. Hence $J' \leq J \leq Z$, a contradiction showing that $J = 0$. Thus we have proved that U is the intersection of all the maximal right-ideals in R which contain U; and it follows from Lemma 3.2 that U is part of every maximal right-ideal in R, since U is, by Theorem 1.2, a nilideal.

4. The anti-radical. We state the following well known fact[14] for future reference.

LEMMA 4.1. *If N is a sum of (a finite or infinite number of) minimal right-ideals in the ring R, then every right-ideal contained in N is a direct summand of N and is itself a direct sum of minimal right-ideals in R.*

The sum[15] $M = M(R)$ of all the minimal right-ideals in the ring R shall be called *the anti-radical of* R; and we put $M(R) = 0$, if there are no minimal right-ideals in R. This definition may be justified by the fact that the upper radical is, under not too narrow assumptions, just the cross-cut of

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[12] The author proved this theorem originally, using a stronger hypothesis. He is indebted to the referee for supplying him with a proof for the theorem in its present form.

[13] Cf. e.g. v. d. Waerden (1), p. 157, Hilfssatz 3.

[14] Cf. e.g. MacLane (1), p. 458, Theorems 3 and 7.

[15] This ideal has been investigated by Hopkins (1) under the hypothesis that the minimum condition be satisfied by the right-ideals in the ring.

all the maximal right-ideals in R (see Theorem 3.3); and by the fact which we shall prove immediately that radical ideals and the anti-radical annihilate each other.

If J is a minimal right-ideal in the ring R, x an element in R, then xJ is either 0 or a minimal right-ideal in R; and this shows that *the anti-radical is a two-sided ideal.*

THEOREM 4.2. *If M is the anti-radical of the ring R and J a nilideal in R, then $MJ = 0$.*

Proof. If b is an element in the minimal right-ideal B, j an element in the nilideal J, and if bj were different from 0, then bj would be an element not 0 in the minimal right-ideal B so that B would be the smallest right-ideal containing bj. Consequently there exists an element r in R such that [16] $b = bjr \pm bj \pm \cdots \pm bj = b(jr \pm j \pm \cdots \pm j) = bj'$ where j' is an element in J, since j is an element in the right-ideal J. There exists an integer i such that $j'^i = 0$ and this leads to the contradiction: $0 \neq bj = bj'j = \cdots = bj'^ij = 0$. We have shown, therefore, that $BJ = 0$ for every minimal right-ideal B in R; and this fact clearly implies $MJ = 0$.

The following condition will be imposed frequently upon the rings under consideration.

(4.A) *If x is an element in the ring R, then x is contained in the right-ideal xR.*

This requirement is met e.g. by all the rings which contain a right-identity element; and (4.A) is satisfied by the ring R if, and only if, $J = JR$ for every right-ideal J in R.

THEOREM 4.3. *If condition (4.A) is satisfied by the ring R, if T is a two-sided ideal in R, and if R/T is a sum of minimal right-ideals, then the anti-radical M contains every element x, satisfying: $xT = 0$.*

Proof. If $xT = 0$ is satisfied by the element x in R, and if the right-ideal J in R contains T and is a minimal right-ideal modulo T, then either $xJ = 0$ or xJ is a minimal right-ideal in R. The right-ideal xR is consequently a sum of minimal right-ideals in R, since R/T is a sum of minimal right-ideals in R/T; and as a sum of minimal right-ideals xR is part of the sum M of all the minimal right-ideals in R. Applying condition (4.A) to $xR \leq M$ we find finally that x is in M whenever $xT = 0$.

[16] This more complicated form for the most general element in the right-ideal generated by bj is due to the fact that no identity element need exist in R.

Remark. The impossibility of omitting condition (4. A) in Theorem 4.3 may be seen from the following example: Denote by R the ring of all the multiples of the prime number p considered modulo p^3 and denote by T the ideal of all the multiples of p^2 in R. It is obvious that T is a two-sided ideal in R such that R/T is a minimal ideal and such that $RT = 0$, though R is not a sum of minimal ideals.

The following statement is an immediate consequence of Theorems 1.2, 4.2 and 4.3.

THEOREM 4.4. (a) $MU = 0$.

(b) *If condition* (4. A) *is satisfied by the ring R, and if R/U is a sum of minimal right-ideals, then the anti-radical is exactly the set of all the the elements x which satisfy: $xU = 0$.*

Remark. The impossibility of omitting the second hypothesis in (b) may be seen from the example of the ring of all the integers whose radical and anti-radical are both 0.

The following lemma will prove useful later on.

LEMMA 4.5. *If the right-ideal J in the ring R is contained in the anti-radical M of R, then $J^2 = J^3$.*

Proof. It is a consequence of Lemma 4.1 that J is a sum of minimal right-ideals. If Z is a minimal right-ideal in R, then Z^2 is a subideal of Z and consequently either $Z^2 = Z$ or $Z^2 = 0$. If Z is a minimal right ideal contained in J, then $Z^2 = Z$ implies that $Z \leq J^2$, and $Z^2 = 0$ implies that Z is part of the cross-cut C of J and the upper radical U. From these facts we deduce that $J = J^2 + C$. From Theorem 4.2 we infer that $JC \leq MU = 0$. Consequently $J^2 = J(J^2 + C) = J^3 + JC = J^3$.

M. Hall has shown [17] that every algebra may be decomposed in one and only one way into the sum of a semi-simple and a "bound" algebra. The following concepts will be needed for an extension of his theorem.

We denote by $A = A(R)$ the set of all the elements x in R which satisfy: $xU = Ux = 0$; and we denote by $B = B(R)$ the set of all the elements b in R which satisfy: $bA^2 = A^2b = 0$. It is obvious that A and B are two-sided ideals in R, and that $U \leq B$.

THEOREM 4.6. *Suppose that condition* (4. A) *is satisfied by the ring R.*

(a) *If R is the direct sum of the two-sided ideals S and T, if $U \leq T$, and if the cross-cut of A and T is part of U, then $S = A^2$ and $T = B$.*

[17] Hall (2), Theorem 2.2.

(b) *If R/U is a sum of minimal right-ideals, then the cross-cut of A^2 and B is 0; and the cross-cut of A and B is part of U.*

(c) *If R/U is a sum of minimal right-ideals, and if the cross-cut of A^2 and the anti-radical M of R is the sum of a finite number of minimal right-ideals,*[18] *then R is the direct sum of A^2 and B.*

Proof. If the two-sided ideals S and T meet the requirements of (a), then $ST = TS = 0$ and consequently $S \leq A$. If D is the cross-cut of A and T, then A is the direct sum of S and D, and D is part of U. Thus $D^2 \leq AU = 0$ and $A^2 = S^2 + D^2 = S$; and this implies $B = T$.

If R/U is the sum of minimal right-ideals, then it follows from Theorem 4.4, (b) that A is part of the anti-radical M of R; and hence it follows from Lemma 4.5 that $A^2 = A^3$. If W is the cross-cut of A^2 and B, then we deduce from Lemma 4.1 the existence of a right-ideal V such that A^2 is the direct sum of W and V. Clearly $A^2W = WA^2 = 0$ and hence we have $A^2 = A^4 = (V + W)^2 = V^2$, proving that $W = 0$. The cross-cut of A and B is part of M and is therefore the sum of minimal right-ideals. If Z is a minimal right-ideal, then either $Z^2 = Z$ or $Z^2 = 0$; and if $Z = Z^2$, then Z is part of A^2 and is therefore not contained in B. If $Z^2 = 0$, then Z is nilpotent and is therefore part of U; and this completes the proof of (b).

If the requirements of (c) are met by the ring R, then A^2 is the sum of a finite number of minimal right-ideals; and since the cross-cut of A^2 and B is 0, it follows from $U \leq B$ that the minimal right-ideals contained in A^2 are idempotent. Now we may deduce by the customary arguments[19] the existence of an idempotent e such that $A^2 = eR$. Denote by E the set of elements x in A^2 such that $xe = 0$. Clearly E is a left-ideal and the two-sided ideal ER satisfies: $(ER)^2 = ERER \leq E^2R \leq EA^2 = EeR = 0$, showing that $ER \leq U$. Applying (4.A) we infer now that $E \leq ER \leq U$; and since the cross-cut of A^2 and U is part of the cross-cut 0 of A^2 and B, it follows that $E = 0$; and from this fact we deduce that $A^2 = eR = eA^2 = Re = A^2e$; and this shows that the element b in R belongs to B if, and only if, $be = eb = 0$. If r is any element in R, then $e(r - ere)$ and $(r - ere)e$ belong both to $eR = Re$ and satisfy therefore: $e(r - ere) = e(r - ere)e = 0$ and $(r - ere)e = e(r - ere)e = 0$. Consequently ere belongs to $A^2 = eRe$ and $r - ere$ belongs to B, i.e. R is the direct sum of A^2 and B.

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[18] It is not known to the author whether or not this hypothesis is needed for the validity of the proposition (c).

[19] E. g. v. d. Waerden (1), pp. 156-158.

5. The anti-radical series. An ascending chain of two-sided ideals $M_v = M_v(R)$ in the ring R may be defined by complete (transfinite) induction as follows:

(i) M_1 is the anti-radical M of R.

(ii) M_{v+1} is the uniquely determined two-sided ideal in R which contains M_v and which satisfies: $M(R/M_v) = M_{v+1}/M_v$.

(iii) If v is a limit-ordinal, then M_v is the join (and therefore the sum) of all the ideals M_u for $u < v$.

(iv) There exists a smallest ordinal $m = m(R)$ such that $M_m = M_{m+1}$.

THEOREM 5.1. *$R = M_m$ for some (finite or infinite) ordinal m if, and only if, the following condition is satisfied by every quotient-ring of R:*

(5. B) *Every right-ideal different from 0 contains a minimal right-ideal.*

Proof. If condition (5. B) is satisfied by every quotient ring of R, and if $M_v < R$, then R/M_v contains a minimal right-ideal so that $M_v < M_{v+1}$. Since $M_m = M_{m+1}$ for some ordinal m, this shows the sufficiency of the condition.

If $R = M_m$ for some ordinal m, and if T is a two-sided ideal different from R, then it is easy to construct a well-ordered ascending chain of right-ideals $J(v)$ with the following properties:

(a) $J(0) = T$.

(b) $J(v) < J(v+1)$ and there does not exist a right-ideal J satisfying: $J(v) < J < J(v+1)$.

(c) If v is a limit-ordinal, then $J(v)$ is the join of all the right-ideals $J(u)$ for $u < v$.

(d) $J(w) = R$ for some (finite or infinite) ordinal w.

If J is some right-ideal in R such that $T < J$, then there exists a smallest ordinal z such that the cross-cut of J and $J(z)$ is different from T, since the cross-cut J of J and $J(w) = R$ is different from T. It is an obvious consequence of (c) that z cannot be a limit-ordinal; and we may readily verify that the cross-cut V of J and $J(z)$ is a minimal right-ideal modulo T, as was to be shown.

COROLLARY 5.2. *If $R = M_m$ for some ordinal m, then the radical K of the ring R exists.*

Proof. If $R = M_m$, then it follows from Theorem 5.1 that (5.B) is satisfied by R/L. Hence it follows from Theorems 1.2 and 3.1 that the lower radical L and the upper radical U of the ring R are equal, i.e. $U = L$ is the radical K of R.

We note that under the hypotheses of Corollary 5.2 every nilideal is contained in the radical K of R.

It should be mentioned finally that (5.B) is satisfied by every quotient ring of R, whenever the minimum condition is satisfied by the right-ideals in R.

6. The powers of the radical. If J is a right-ideal in the ring R, then the (finite and transfinite) powers J^v of J are defined by transfinite induction as follows:

(i) $J^1 = J$.

(ii) $J^{v+1} = JJ^v$.

(iii) If v is a limit-ordinal, then J^v is the cross-cut of all the J^u for $u < v$.

(iv) There exists a (smallest) ordinal $s = s(J)$ such that $J^s = J^{s+1}$.

It is clear that the powers of a right-ideal are right-ideals; and that the powers of a two-sided ideal are two-sided ideals.

If J is a right-ideal in R, and if $u \leq v$, then $J^v \leq J^u$.

Proof. We prove this contention by complete induction with regard to v. It is certainly true for $v = 1$; and thus we may assume that $J^w \leq J^u$ for $u \leq w < v$.

Case 1. v is a limit-ordinal.

Then J^v is the cross-cut of all the J^u for $u < v$ so that $u < v$ implies $J^v \leq J^u$.

Case 2. $v = w + 1$ for w a limit-ordinal.

Then $J^v = JJ^w \leq JJ^u = J^{u+1}$ for every $u \leq w$. Thus J^v is part of the cross-cut of all the J^{u+1} for $u < w$; and this cross-cut is just J^w, since w is a limit-ordinal.

Case 3. $v = z + 2$ for some ordinal z.

Then $J^v = JJ^{z+1} \leq JJ^z = J^{z+1} \leq J^w$ for $w \leq z + 1 < v$; and this completes the proof.

THEOREM 6.1. *If J is a nilideal, then $M_v J^v = 0$ for every ordinal v.*

Proof. It is a consequence of Theorem 4. 2 that $M_1 J^1 = MJ = 0$. Thus we may assume that our assertion holds for every $u < v$.

Case 1. $v = w + 1$ is not a limit-ordinal.

Then M_v/M_w is the anti-radical of R/M_w and $(J + M_w)/M_w$ is a nilideal in the quotient ring R/M_w. Hence it follows from Theorem 4. 2 that their product is 0; and from this fact we deduce $M_v J \leq M_w$. From the induction-hypothesis we infer $M_w J^w = 0$. Consequently we find that $M_v J^v = M_v J J^w \leq M_u J^w = 0$.

Case 2. v is a limit-ordinal.

If x is any element in M_v, then we deduce from the definition of M_v as the join of the M_u with $u < v$ the existence of an ordinal $u < v$ such that x is an element in M_u. It is a consequence of the definition of J^v that $J^v \leq J^u$; and hence it follows from the induction-hypothesis that $xJ^v \leq M_u J^u = 0$; and thus we have shown that $M_v J^v = 0$.

COROLLARY 6. 2. *If $R = M_m$ for some ordinal m, then $K^{m+1} = 0$.*

Remark. The existence of the radical K of R is assured by Corollary 5. 2.

Proof. It is an immediate consequence of Theorem 6. 1 that

$$K^{m+1} = KK^m \leq RK^m = M_m K^m = 0.$$

Note that we could infer $K^m = 0$ from $R = M_m$, if 0 were the only element x in R satisfying $Rx = 0$.

That $K^k = 0$ need not imply $R = M_m$, may be seen from the example of the ring of all the integers where $M_v = K^v = 0$ for every v.

THEOREM 6. 3. *If $R = M_m$ for some ordinal m, then each of the following properties of the right-ideal J in R implies all the others:*

(i) *J is a nilideal.*

(ii) *J is a part of the radical K of R (whose existence is assured by Corollary 5. 2).*

(iii) *$J^k = 0$ for some ordinal k.*

Proof. It is a consequence of Theorems 5. 1 and 3. 1 that every nilideal is part of the radical. If the right-ideal J is part of the radical K of R, then we deduce from Corollary 6. 2 that $J^{m+1} \leq K^{m+1} = 0$. If finally $J^k = 0$ for some ordinal k, then we denote by J_v the cross-cut of J and of M_v, and we put $J_0 = 0$. We proceed to prove by complete (transfinite) induction that

every J_v is a nilideal, a fact that is clearly true for $v = 0$. Hence we may assume that J_u is a nilideal for every $u < v$.

Case 1. $v = w + 1$ is not a limit-ordinal.

The right-ideal $J^* = (M_w + J_v)/M_w$ is part of the anti-radical M_v/M_w of the ring R/M_w; it is, therefore, the sum of minimal right ideals in R/M_w. If Z^* is a minimal right-ideal in R/M_w, then either $Z^{*2} = 0$ or $Z^* = Z^{*2}$. If the idempotent minimal right-ideal Z^* were part of J^*, then there would exist an element e in J_v which is not contained in J_w, though $e - e^2$ is an element in J_w, since every idempotent minimal right-ideal contains an idempotent not 0.[20] It is a consequence of the induction-hypothesis that J_w is a nilideal and that therefore $e - e^2$ is a nilelement. Consequently we are able to deduce from a theorem of G. Köthe [21] the existence of an idempotent $j \neq 0$ in $J_v \leq J$, a fact that is clearly incompatible with $J^k = 0$. Thus we have shown that J^* is the sum of minimal right-ideals whose squares are 0; and from this result it is easily deduced that $J^{*2} = 0$. But $J_v^2 \leq M_w$ is an immediate consequence of $J^{*2} = 0$. Since M_w is a nilideal, the square of every element in J_v is a nilelement, i. e. J_v is a nilideal.

Case 2. v is a limit-ordinal.

Then every element in J_v is contained in some J_u for $u < v$; and it follows from the induction-hypothesis that every element in J_v is a nilelement, i. e. that J_v is a nilideal.

This shows that J is a nilideal, since $J = J_m$ is a consequence of $R = M_m$.

Remark. If J is an ideal, neither 0 nor 1, in the ring of natural integers, then $J^\omega = 0$, though J is not a nilideal; and this shows that the hypothesis $R = M_m$ cannot be omitted in Theorem 6. 3.

THEOREM 6. 4. *If condition* (4. A) *is satisfied by the ring* R, *and if* R/U *is a sum of minimal right-ideals, then* M_i *is, for every positive integer* i, *exactly the set of all the elements* x *in* R *which satisfy:* $xU^i = 0$.

Proof. The validity of our contention for $i = 1$ is an immediate consequence of Theorem 4. 4, (b). Thus we may assume that M_{i-1} is exactly the set of all the elements x in R which satisfy: $xU^{i-1} = 0$. The ideal M_i/M_{i-1} is the anti-radical of the ring R/M_{i-1}; and $(U + M_{i-1})/M_{i-1}$ is, by Theorem 1. 2, a two-sided nilideal in R/M_{i-1} modulo which this ring is a sum of

[20] E. g. v. d. Waerden (1), p. 157, Hilfssatz 3.
[21] Köthe (1), p. 168, Hilfssatz 3.

minimal right-ideals. Hence it follows from Theorems 4.2 and 4.3 that M_i is exactly the set of all the elements x in R which satisfy: $xU \leq M_{i-1}$. But it is a consequence of the induction-hypothesis that xU is part of M_{i-1} if, and only if, $0 = xUU^{i-1} = xU^i$; and x is therefore an element in M_i if, and only if, $xU^i = 0$, as was to be shown.

THEOREM 6.5. *If the right-ideal J in R is part of M_i for i a positive integer, then $J^{2^i} = J^{2^{i+1}}$.*

Proof. It is a consequence of Lemma 4.5 that our theorem is true for $i = 1$. Thus we may assume the validity of the theorem for subideals of M_i and we have to derive it from this induction hypothesis for the subideals of M_{i+1}.

Assume now that the right-ideal J in R is part of M_{i+1}. The right-ideal $(M_i + J)/M_i$ in R/M_i is part of the anti-radical $M(R/M_i) = M_{i+1}/M_i$ of the quotient ring R/M_i; and hence we deduce from Lemma 4.5 that $M_i + J^2 = M_i + J^3$. The cross-cut C of M_i and J^2 is a subideal of M_i and consequently we may infer from the induction hypothesis that $C^{2^i} = C^{2^{i+1}}$, and the above equation may be restated as $J^2 = C + J^3$, since $J^3 \leq J^2$ and since we may apply the modular (Dedekind's) law. Expanding $(C + J^3)^{2^{i-1}}$ we obtain

$$J^{2^{i+1}-2} = (C + J^3)^{2^{i-1}} = C^{2^{i-1}} + \sum_{j=1}^{2^{i-1}} V_j$$

where every summand V_j is the sum of products of $2^i - 1$ ideals of which j are equal to J^3 and the remaining $2^i - 1 - j$ factors are equal to C. Since $C \leq J^2$, it follows that $V_j \leq J^{3j+2(2^i-1-j)} = J^{2^{i+1}+j-2} \leq J^{2^{i+1}-1}$ or

$$CV_j \leq J^{2+2^{i+1}-1} = J^{2^{i+1}+1}.$$

Consequently we find that

$$\begin{aligned} J^{2^{i+1}} = J^2 J^{2^{i+1}-2} &= (C + J^3) J^{2^{i+1}-2} = C(C^{2^{i-1}} + \sum_{j=1}^{2^{i-1}} V_j) + J^{2^{i+1}+1} \\ &= C^{2^i} + J^{2^{i+1}+1} = C^{2^{i+1}} + J^{2^{i+1}+1} \\ &= J^{2^{i+1}+1}, \end{aligned}$$

since $C^{2^{i+1}} \leq J^{2^{i+1}+2} \leq J^{2^{i+1}+1}$; and this completes the proof.

7. Nilpotent ideals. It is an immediate consequence of the definition of radical ideals (cf. condition (1.c)!) that they contain every nilpotent right-ideal. On the other hand it has been pointed out that radical ideals need not be nilpotent. Thus we shall give in this section several criteria for a nilideal, in particular a radical ideal, to be nilpotent.

THEOREM 7.1. *Every nilideal contained in M_i for i a positive integer is nilpotent.*

Proof. It is a consequence of Theorem 6.1 that a nilideal J contained in M_i satisfies: $J^{i+1} = JJ^i \leq M_iJ^i = 0$, and J is therefore nilpotent.

THEOREM 7.2. *If the maximum-condition*[22] *is satisfied by the nilpotent right-ideals in the ring R, then the lower radical of R is nilpotent.*

Remark. That the hypothesis of this theorem is not sufficient for proving that the upper radical is nilpotent, may be seen from the example in section **2**. It seems to be an open question whether or not the maximum condition for right-ideals is sufficient for nilpotence of the upper radical.

Proof. There exists a greatest nilpotent right-ideal G in R. If J is a nilpotent right-ideal in R, then $G + J$ is a nilpotent right-ideal so that J is part of G. Consequently G is the sum of all the nilpotent right-ideals in R. It is a consequence of Lemma 1.1 that G is a two-sided ideal in R. There cannot exist nilpotent right-ideals different from 0 in R/G, since $Z^i \leq G$ and the nilpotence of G imply the nilpotence of Z. This shows that the nilpotent ideal G is a radical ideal; and it is readily verified that G is the lower radical of the ring R.

THEOREM 7.3. *If $R = M_m$ for some ordinal m, and if at least one of the two chains M_v and K^v is finite, then the radical K of R is nilpotent.*

Remark. The existence of the radical K is a consequence of Corollary 5.2.

Proof. If $R = M_m$ for some finite ordinal m, then the nilpotence of the radical is a consequence of Theorem 7.1, since the radical is by Theorem 1.2 a nilideal.

If $R = M_m$ for some ordinal m, then it follows from Corollary 6.2 that $K^{m+1} = 0$. If the chain K^v is finite, then there exists a finite ordinal i such that $K^i = K^{i+1} = \cdots = K^{m+1} = 0$.

The next theorem is a partial converse of Theorem 7.3.

THEOREM 7.4. *If condition* (4.A) *is satisfied by the ring R, if the radical K of R exists and is nilpotent, and if R/K is a sum of minimal right-ideals, then $R = M_i$ for some positive integer i.*

Proof. There exists a positive integer i such that $K^i = 0$; and it is a

[22] It states that every not vacuous class of nilpotent right-ideals contains at least one greatest ideal.

consequence of Theorem 6.4 that M_i is exactly the set of all the elements x in R which satisfy: $xK^i = 0$. Since $RK^i = R0 = 0$, it follows that $M_i = R$.

The third condition in Theorem 7.4 cannot be omitted, since in the ring of all the integers radical and anti-radical are both 0.

COROLLARY 7.5. *Suppose that condition* (4.A) *is satisfied by the ring R, that R/U is a sum of minimal right-ideals, and that 0 is the only element x in R that satisfies $Rx = 0$. Then*

$$U^i = 0 \text{ if, and only if, } M_i = R \text{ (for } i \text{ a positive integer)}.$$

Proof. That $U^i = 0$ implies $M_i = R$, is an immediate consequence of Theorem 6.4; and that $M_i = R$ implies $U^i = 0$, may be inferred from Theorem 6.1, since $0 = M_iU^i = RU^i$, and since $S = 0$ is a consequence of $RS = 0$.

8. Maximum and minimum conditions.[23] In the results of the previous sections there occurred hypotheses that are connected in various ways with maximum and minimum conditions. In this section we investigate the relations between these properties.

THEOREM 8.1. *Suppose that the maximum condition is satisfied by the right-ideals in the ring R. Then the minimum condition is satisfied by the right-ideals in R if* (*and only if*) *condition* (5.B) *is satisfied by every quotient-ring of R.*

Proof. If condition (5.B) is satisfied by every quotient ring R, then there exists, by Theorem 5.1, an ordinal m such that $R = M_m$. Since the ideals M_v form an ascending chain, and since the maximum condition is satisfied by the right-ideals in R, there exists a positive integer i such that $M_i = M_{i+1}$. Since $R = M_m$ for some ordinal m, it follows that $R = M_i$ for i a finite ordinal. It is a consequence of the definition of the series M_v that M_{v+1}/M_v is a sum of minimal right-ideals in R/M_v; and we infer from the maximum condition for right-ideals that M_{v+1}/M_v is a sum of a finite number of minimal right-ideals. Consequently there exists a finite composition series[24] of right-ideals in R; and it is well known that the minimum condition for right-ideals is a consequence of this fact.

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[23] The maximum (minimum) condition is satisfied by the right-ideals in the ring R, if in every not vacuous set of right-ideals there exists a right-ideal which is not smaller (greater) than any other right-ideal in the set.

[24] A composition series is a densest finite ascending chain of right-ideals.

If there exists in the ring R an infinite independent set [25] of right-ideals, then there exists in R a countably infinite, independent set of right-ideals $J(i) \neq 0$ for $i = 1, 2, \cdots$. The chain of right-ideals $\sum_{j<i} J(j)$ is an infinite ascending chain of right-ideals; and the chain $\sum_{i<j} J(j)$ is an infinite descending chain of right-ideals. Thus the maximum as well as the minimum condition for right-ideals implies the following property of rings R:

(8. C) *There does not exist an infinite independent set of right-ideals.*

If (8. C) is satisfied by the ring R, then it follows from Lemma 4. 1 that the anti-radical M of R is the (direct) sum of a finite number of minimal right-ideals. The following statement is a partial converse of this fact.

THEOREM 8. 2. *If condition* (5. B) *is satisfied by the ring R, and if the anti-radical M is a sum of a finite number of minimal right-ideals in R, then* (8. C) *is satisfied by R.*

Proof. Suppose that S is an independent set of right-ideals different from 0 in R. If J is a right-ideal in the set S, then J contains a minimal right-ideal J'. The set S' of these minimal right-ideals J' is independent and contains as many elements as S. Since there does not exist an infinite independent set of minimal right-ideals in R, it follows that S is a finite set and that therefore (8. C) is satisfied by R.

LEMMA 8. 3. *If conditions* (5. B) *and* (8. C) *are satisfied by the ring R, and if R does not contain nilpotent right-ideals different from* 0, *then R is the sum of a finite number of minimal right-ideals.*

Remark. This is a generalization of the so-called " Fundamental Theorem on Semi-simple Rings." [26]

Proof. Suppose that we have constructed idempotents $e_1, \cdots, e_n$ meeting the following requirements:

(a) $e_i R$ is a minimal right-ideal.

(b) $e_i e_j = 0$ for $i \neq j$.

This is certainly possible for $n = 0$.

We note that the right-ideals $e_i R$ form an independent set of right-ideals.

[25] The set S of ideals is said to be *independent*, if the cross-cut of any ideal J in S with the sum of the other ideals in S is 0.

[26] Cf. e. g. v. d. Waerden (1), p. 156.

Let $W(n)$ be the right-ideal of all the elements x satisfying: $\sum_{i=1}^{n} e_i x = 0$ ($W(0) = R$). Then it is readily seen that $R = W(n) + \sum_{i=1}^{n} e_i R$; and we have effected the proof of our lemma, if $W(n) = 0$. Should $W(n)$ be different from 0, then it contains a minimal right-ideal J; and J contains, by a well known theorem,[27] an idempotent $e \neq 0$, since $J = J^2$. Put $e_{n+1} = e - e\sum_{i=1}^{n} e_i$. It is readily verified that e_{n+1} is an idempotent, satisfying $e_i e_{n+1} = e_{n+1} e_i = 0$ for $i = 1, \cdots, n$, and that $J = e_{n+1} R$; and thus the idempotents $e_1, \cdots, e_n, e_{n+1}$ meet the requirements (a) and (b).

If this construction would never stop, i. e. if the right-ideal $W(n)$ were different from 0 for every positive n, then we would be led to the infinite independent set of minimal right-ideals $e_1 R, \cdots, e_i R, \cdots$, contradicting condition (8. C). This completes the proof.

THEOREM 8. 4. *If condition* (4. A) *is satisfied by the ring R, then each of the following properties implies all the others:*

(1) *The maximum and the minimum condition are satisfied by the right-ideals in R.*

(2) *The minimum condition is satisfied by the right-ideals in R.*

(3) *Conditions* (5. B) *and* (8. C) *are satisfied by every quotient-ring of the ring R; and the descending chain of the powers of the radical*[28] *K of R is finite.*

Proof. It is obvious that (1) implies (2); and that (3) is a consequence of (2) is a consequence of facts we mentioned when introducing condition (8. C).

If (3) is satisfied by the ring R, then it follows from Theorem 5. 1 that $R = M_m$ for some ordinal m; and from Lemma 8. 3 that R/K is the sum of a finite number of right-ideals. There exists a positive integer i such that $K^i = K^{i+1}$; and it is a consequence of Theorem 6. 4 that M_n is, for finite n, exactly the set of all the elements x, satisfying $xK^n = 0$. This shows the equality of $M_i = M_{i+1}$. Hence we have $M_i = M_{i+1} = \cdots = M_m = R$. Since there do not exist infinite independent sets of right-ideals in R/M_n, it follows from Lemma 3. 3 that M_{n+1}/M_n is the sum of a finite number of minimal right-ideals. Consequently there exists a finite composition series of right-ideals in R, a fact which is equivalent to our property (1).

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[27] Cf. e. g. v. d. Waerden (1), p. 157.

[28] The existence of the radical K is assured by Theorem 5. 1 and Corollary 5. 2.

Remarks. 1. In proving that (1) is a consequence of (3) we did not use condition (3) in its entirety. That condition (5. B) is satisfied by every quotient ring of R, and that the powers of the radical form a finite chain, are hypotheses indispensable for the above proof. But it is not necessary to assume that (8. C) is satisfied by every quotient ring of R. It would have been sufficient to make sure that the quotient rings R/M_i for finite i and R/K meet the requirement (8. C); and for the latter assumption we could have substituted the weaker hypothesis that R/K is a sum of minimal right-ideals.

2. It has been shown elsewhere [29] that a ring with minimum-condition for right-ideals possesses a right-identity element if, and only if, it satisfies condition (4. A); and Ch. Hopkins [30] has shown that the maximum condition for right-ideals is a consequence of the minimum condition for right-ideals, provided there exists a right-identity element. It should be noted, however, that condition (4. A) is indispensable for the validity of this Theorem 8. 4, as may be seen from the following example: denote by R any infinite abelian group without elements of infinite order which contains only a finite number of elements of order a prime; such a group is the direct sum of a finite abelian group and of a finite number of groups of type [31] p^∞. If we define $xy = 0$ for every pair of elements x and y in R, then R is a commutative ring, satisfying $0 = R^2$. The ideals in R are just the subgroups of the additive group R; and thus it becomes apparent that the minimum condition is satisfied by the ideals in R, but not the maximum condition.

9. Existence of the identity. The following statement is basic for the considerations of this section.

LEMMA 9. 1. *If conditions* (5. B) *and* (8. C) *are satisfied by* [32] R/K, *then there exists an idempotent* e *in* R *such that* $x \equiv ex \equiv xe$ modulo K *for every element* x *in* R.

Proof. R/K is a ring without nilpotent right-ideals different from 0, by Theorem 1. 2; and R/K is, by Lemma 8. 3, the sum of a finite number of minimal right-ideals. Hence there exists, by a well known theorem, [33] an identity element in R/K, i. e. there exists an element f in R such that

[29] Baer (1), Corollary to Theorem 6.

[30] Hopkins (1), p. 726, Theorem 6. 4.

[31] The groups of type $p\infty$ have been discovered by H. Prüfer; they are generated by a countable number of elements g_i subject to the relations: g_0 is an element of order p; $g_{i-1} = g_i p$.

[32] The existence of the radical K is assured by Theorem 5. 1 and Corollary 5. 2.

[33] Cf. e. g. v. d. Waerden (1), p. 156.

$x \equiv xf \equiv fx$ modulo K for every x in R. If f is any element meeting this requirement, then $f^2 - f = f'$ is an element in K; and we deduce from Theorem 1.2 the existence of a positive integer $n = n(f)$ such that $f'^n = 0$. Thus there exists among the elements which represent an identity element modulo K one, say e, with minimal $n(e)$. It is readily verified that e is an idempotent (which clearly meets all our requirements), since otherwise [34] $e_1 = e - 2ee' + e'$ [for $e' = e^2 - e$] would be an element which represents the identity element modulo K, though $n(e_1)$ were smaller than $n(e)$.

THEOREM 9.2. *If $R = M_m$ for some (finite or infinite) ordinal m, and if condition* (8. C) *is satisfied by the ring* [35] *R/K, then the property that every element x in R is contained in Rx is a necessary and sufficient condition for the existence of a left-identity-element in R.*

Proof. If e is a left-identity-element in R, then $x = ex$ belongs to Rx, showing the necessity of the condition. We assume now that the condition: "x belongs to Rx" is satisfied by the ring R. It is a consequence of Theorem 5.1 that condition (5. B) is satisfied by every quotient ring of R; and hence we may deduce from Lemma 9.1 the existence of an idempotent e such that $x \equiv ex \equiv xe$ modulo K for every element x in R. We denote by W the set of all the elements x in R such that $ex = 0$. Since $x \equiv ex$ modulo K, it follows that W is part of K. Finally we have $R = Re + K$.

We proceed to prove by complete (transfinite) induction that $W \leq K^v$ for every v. This is certainly true for $v = 1$; and thus we may assume that $W \leq K^u$ for every $u < v$.

Case 1. $v = w + 1$ is not a limit-ordinal.

Then we deduce from the hypothesis that x is an element in Rx and from the induction hypothesis the inequality:

$$W \leq RW = ReW + KW = KW \leq KK^w = K^v.$$

Case 2. v is a limit-ordinal.

Then K^v is the cross-cut of all the K^u for $u < v$; and $W \leq K^v$ is an immediate inference from the induction-hypothesis.

It is a consequence of $R = M_m$ and Corollary 6.2 that $K^{m+1} = 0$. Thus W, as a part of K^{m+1}, is 0.

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[34] This construction is due to Köthe (1), p. 169 and Dickson (1), p. 123.

[35] The existence of the radical K is assured by Theorem 5.1 and Corollary 5.2.

Since $x - ex$ belongs to W for every x in R, it follows that $x = ex$ for every x in R, i. e. that e is a left-identity-element in R.

Remark. The condition: $R = M_m$ was only needed in the proof to assure that condition (5. B) is satisfied by R/K and that $K^{m+1} = 0$; and these two apparently weaker conditions may be substituted for $R = M_m$.

The *identity* is an element e in R that satisfies $ex = xe = x$.

COROLLARY 9. 3. *If $R = M_m$ for some (finite or infinite) ordinal m, and if condition (8. C) is satisfied by the ring*[35] *R/K, then the following two conditions are necessary and sufficient for the existence of the identity 1 in the ring R:*

(i) *$xR = 0$ implies $x = 0$; and*

(ii) *Rx contains x.*

Proof. The necessity of the conditions (i) and (ii) is obvious. If these conditions are satisfied by the ring R, then we deduce from Theorem 9. 2 the existence of a left-identity-element e in R; and e is the identity 1 in R, since

$$(x - xe)R = (x - xe)eR = (xe - xe)R = 0 \quad \text{implies} \quad x = xe \quad \text{by} \quad \text{(i)}.$$

THEOREM 9. 4. *If $R = M_m$ for some (finite or infinite) ordinal m, and if condition (8. C) is satisfied by the ring*[36] *R/K, then condition (4. A) is a necessary and sufficient condition for the existence of a right-identity-element in R.*

Proof. If e is a right-identity-element in R, then $x = xe$ is contained in the right-ideal xR, showing the necessity of condition (4. A). If condition (4. A) is satisfied by the ring R, then we deduce from Theorem 5. 1 the validity of condition (5. B) in every quotient ring of R; and hence we may infer from Lemma 9. 1 the existence of an idempotent e satisfying $x \equiv ex \equiv xe \bmod K$ for every element x in R. Clearly $R = eR + K$.

It will be convenient to put $0 = M_0$. Then we prove by complete (transfinite) induction that $M_v = M_v e$ and that 0 is the only element x in M_v satisfying $xe = 0$. This fact is patently true for $v = 0$ and thus we assume it to be true for every $u < v$.

Case 1. $v = w + 1$ is not a limit-ordinal.

Then $M_w = M_w e$ and 0 is the only element x in M_w such that $xe = 0$.

[36] The existence of the radical K is assured by Theorem 5. 1 and Corollary 5. 2.

If y is an element in M_v such that $ye = 0$, then $yR = yeR + yK = yK$; and since M_v/M_w is the anti-radical of the ring R/M_w, it follows from Theorem 4.4, (a) that $yR = yK \leq M_w$. But it is a consequence of (4.A) that y is an element in yR and therefore in M_w; and now we deduce from the induction-hypothesis that $y = 0$. Consequently 0 is the only element x in M_v such that $xe = 0$; and $M_v = M_ve$ is an obvious consequence of this fact.

Case 2. v is a limit-ordinal.

If x is an element in M_v, then there exists an ordinal $u < v$ such that x is an element in M_u. Hence it follows from the induction-hypothesis that $x = xe$ showing that $M_v = M_ve$ and that therefore 0 is the only element z in M_v such that $ze = 0$.

Now it is evident that $R = M_m = M_me = Re$ and that consequently e is a right-identity-element for R.

By essentially the same arguments as the ones used in the proof of Corollary 9.3 we deduce the following statement from Theorem 9.4.

COROLLARY 9.5. *If $R = M_m$ for some (finite or infinite) ordinal m, and if condition* (8.C) *is satisfied by the ring*[36] *R/K, then the following two conditions are necessary and sufficient for the existence of the identity* 1 *in the ring R:*

(i) *$Rx = 0$ implies $x = 0$; and*

(ii) *xR contains x.*

It has been assumed in Theorems 9.2 and 9.4 and in Corollaries 9.3 and 9.5 that condition (8.C) is satisfied by the ring R/K. The impossibility of omitting this hypothesis may be seen from the following example: Denote by F a field, by G an abelian group which admits the elements in F as operators and whose rank over F is infinite, and by R the ring of all the (proper and improper) automorphisms of the group G over F which map G upon a subgroup of finite rank. It is readily seen that R contains neither a left-identity-element nor a right-identity-element, that $R = M$ and that $K = 0$. If r is any element in R, then denote by G_r the set of all the elements in G which are mapped upon 0 by r. Clearly G^r and G/G_r are isomorphic groups of finite rank over R; and hence there exist idempotents e, f in R such that every element in G^r is left invariant by e, G_r is mapped upon 0 by f and every coset of G/G_r is mapped by f upon an element in itself, since both G^r and G_r are direct summands of G. Clearly $r = re = fr$ showing that every subset S of R is contained in both SR and RS.

THEOREM 9.6. *If conditions* (5.B) *and* (8.C) *are satisfied by*[36] R/K, *then the following pair of properties is a necessary and sufficient condition for the existence of a left-identity-element in* R:

(a) *Every right-ideal different from* R *is contained in a maximal right-ideal in* R.

(b)[37] *The radical* K *of* R *is the cross-cut of all the maximal right-ideals in* R.

Proof. Suppose that there exists a left-identity-element e in R. If the right-ideal J in R is different from R, then there exists a greatest right-ideal G in R which contains J, but which does not contain e. Clearly G is a maximal right-ideal in R, since $R = eR$. It is a consequence of Theorem 1.2 and of Lemma 8.3 that R/K is a sum of minimal right-ideals; and hence it may be inferred from Theorem 3.4 that K is the cross-cut of all the maximal right-ideals in R.

Suppose now, conversely, that the conditions (a) and (b) are satisfied by the ring R. There exists by Lemma 9.1 an idempotent e in R such that $x \equiv ex \equiv xe$ modulo K for every element x in R. Since the elements v satisfying $ev = 0$ are certainly contained in K, it follows that $R = eR + K$. If the right-ideal eR in R were different from R, then there would exist a maximal right-ideal G in R which contains eR. It is a consequence of (b) that $K \leq G$; and thus we are led to the contradiction: $R = eR + K \leq G < R$. Hence $R = eR$ and e is a left-identity-element in R.

Remarks. 1. If R is the ring of all the even rational integers, then the radical of R is 0; and it is readily seen that conditions (a) and (b) are satisfied by R. Condition (8.C) is satisfied too; but R does not contain an identity. This shows the impossibility of omitting the hypothesis that condition (5.B) be satisfied by R/K.

2. Suppose that the abelian group R is the direct sum of a group K of type 2^∞ and of a cyclic group of order 2 which is generated by an element e. In R we define a commutative multiplication by the rules: $uv = 0$, if at least one of the factors u and v is in K; and $e = e^2$. It is readily seen that K is the only maximal ideal in R, and that K is the radical in R. Thus conditions (5.B) and (8.C) are satisfied by R/K and condition (b) is satisfied by R. But there does not exist an identity element in R; and this shows the impossibility of omitting condition (a) in Theorem 9.6.

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[37] Considering this condition (b) and Theorem 4.4 it may be shown that Theorem 9.6 is a generalization of a theorem due to M. Hall; cf. Hall (1), p. 362, Theorem 3.2.

10. The quasi-regular elements. S. Perlis [38] has recently discovered a characterization of the radical of an algebra which is rather different from the usual criteria. We devote this section to a generalization of his theory.

An element x in the ring R is termed *quasi-regular,*[39] if there exists an element y in R satisfying: $x + y + xy = 0$. If the ring R possesses an identity element 1, then a necessary and sufficient condition for quasi-regularity of the element x in R is the existence of a solution z of the equation: $(1 + x)z = 1$.

The *right-ideal* J in the ring R shall be termed *quasi-regular,* if every element in J is quasi-regular.

THEOREM 10.1. *If z is an element in the quasi-regular right-ideal J, then there exists one and only one solution $x = z'$ of the equation $z + x + zx = 0$; and such a solution satisfies $zz' = z'z$.*

Proof. If z is an element in the quasi-regular right-ideal J, then there exists an element y in R such that $z + y + zy = 0$. If y is some solution of this equation, then $y = -(z + zy)$ is an element in J, since z is an element in the right-ideal J. Since J is quasi-regular, so is y; and hence there exists an element w in R such that $y + w + yw = 0$. Consequently we find that

$$\begin{aligned} z &= z + y + w + yw + z(y + w + yw) \\ &= z + y + zy + (y + z + zy)w + w = w; \end{aligned}$$

and this shows that $0 = y + w + yw = y + z + yz$; and we deduce $zy = yz$ from $y + z + zy = 0$.

Suppose now that $x = y$ and $x = y'$ are solutions of the equation: $z + x + zx = 0$. From what we have shown in the previous paragraph of the proof, it follows that $0 = z + y + yz$; and thus we find that

$$\begin{aligned} y' &= y' + z + y + yz + (z + y + yz)y' \\ &= z + y' + zy' + y(z + y' + zy') + y = y; \end{aligned}$$

and this completes the proof.

We denote by $S = S(R)$ *the sum of all the quasi-regular right-ideals in R.*

THEOREM 10.2. *$S(R)$ is a quasi-regular right-ideal in the ring R.*

Proof. It is readily seen that it suffices to prove the following statement: If u is contained in a quasi-regular right-ideal, and if v is contained in a quasi-regular right-ideal, then $u + v$ is an element of some quasi-regular right-ideal.

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[38] Perlis (1).

[39] Perlis (1), p. 129.

We note that (in the absence of identity elements) the right-ideal generated by the element w consists of all the elements of the form: $wi + wr$ for r in R and i an ordinary rational integer (i need not be an element in R). Thus we have to show that $(u+v)i + (u+v)r$ is quasi-regular for every r in R and for every rational integer i, whenever both u and v are contained in quasi-regular right-ideals.

Since $ur + ui$ belongs to the right-ideal generated by u, it is quasi-regular; and hence there exists an element s such that

$$(ur + ui) + s + (ur + ui)s = 0;$$

and since $v(r + rs + ri) + vi$ belongs to the right-ideal generated by v, it is quasi-regular too; and this assures the existence of an element t satisfying:

$$v(r + rs + si) + vi + t + (v(r + rs + si) + vi)t = 0.$$

Hence

$$\begin{aligned}((u+v)r + (u+v)i) + (s + t + st) + ((u+v)r + (u+v)i)(s + t + st)\\ = ur + ui + s + (ur + ui)s + (ur + ui + s + (ur + ui)s)t\\ + v(r + rs + si) + vi + t + (v(r + rs + si) + vi)t = 0;\end{aligned}$$

and this proves that $(u+v)r + (u+v)i$ is quasi-regular for every r in R and for every natural integer i; i. e. $u + v$ is contained in a quasi-regular right-ideal in R, as was to be shown.

THEOREM 10. 3. *If the right-ideal J in R consists of elements that are quasi-regular modulo* [40] *$S(R)$, then $J \leq S(R)$.*

Proof. If the element x in R is quasi-regular modulo $S(R)$, then there exists an element y such that $x + y + xy$ is in $S(R)$; and it follows from Theorem 10. 2 that $x + y + xy$ is quasi-regular. Consequently there exists an element z in R such that

$$0 = x + y + xy + z + (x + y + xy)z = x + (y + z + yz) + x(y + z + yz);$$

and x is therefore a quasi-regular element. The right-ideal J is thus quasi-regular, if each of its elements is quasi-regular modulo $S(R)$.

COROLLARY 10. 4. *If the right-ideal J in R is a nilideal modulo* [41] *$S(R)$, then $J \leq S(R)$.*

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[40] I. e. to every element j in J there exists an element h in R such that $j + h + jh$ belongs to $S(R)$.

[41] I. e. to every element j in J there exists a positive integer n such that j^n belongs to $S(R)$.

Proof. If the element w is a nilelement modulo $S(R)$, then there exists a positive integer n such that w^{2n+1} is an element in $S(R)$. Since

$$w + \sum_{i=1}^{2n} (-1)^i w^i + w \sum_{i=1}^{2n} (-1)^i w^i = w^{2n+1},$$

it follows that w is quasi-regular modulo $S(R)$; and this shows that our contention is an immediate consequence of Theorem 10. 3.

LEMMA 10. 5. *If the element e in $S(R)$ is an idempotent modulo the two-sided ideal T in R, then e belongs to T.*

Proof. If e is an element in $S(R)$, then $-e$ is quasi-regular. Hence there exists an element f in R such that $-e+f-ef=0$. If e is furthermore an idempotent modulo the two-sided ideal T, then

$$e \equiv e^2 \equiv e(f-ef) \equiv ef - e^2f \equiv ef - ef \equiv 0 \bmod T,$$

i. e e belongs to T.

THEOREM 10. 6. *If the two-sided ideal T in R is part of $S(R)$, if every right-ideal, not 0, in R/T contains a minimal right-ideal, and if 0 is the only nilpotent right-ideal in R/T, then $T=S(R)$.*

Proof. If T were different from $S(R)$, then there would exist a right-ideal J between T and $S(R)$ such that J/T is a minimal right-ideal in R/T. Since 0 is the only nilpotent right-ideal in R/T, it follows that $J/T=(J/T)^2$; and hence we may deduce from known theorems[42] that J/T contains an idempotent different from 0. But it follows from Lemma 10. 5 that this is impossible; and this shows that $T=S(R)$.

COROLLARY 10. 7. *If every right-ideal, not 0, in R/U contains a minimal right-ideal, then $S(R)$ is the upper radical U of R.*[43]

Proof. It is a consequence of Theorem 1. 2 that U is a nilideal and that R/U does not contain nilpotent right-ideals different from 0; it is a consequence of Corollary 10. 4 that U is part of $S(R)$; and hence we may deduce from Theorem 10. 6 that $U=S(R)$.

We denote by $S^*=S^*(R)$ *the sum of all the quasi-regular two-sided ideals in R.* Clearly S^* is a two-sided ideal which is part of $S(R)$; and hence it follows from Theorem 10. 2 that $S^*(R)$ is quasi-regular. It is a consequence of Theorem 10. 3 that $S^*(R/S^*(R))=0$.

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[42] Cf. e. g. v. d. Waerden (1), p. 157, Hilfssatz 3.

[43] This is readily seen to be an extension of Perlis' (1) Theorems 1 and 2.

COROLLARY 10.8. *If every right-ideal, not* 0, *in* $R/S^*(R)$ *contains a minimal right-ideal, then* $S^*(R) = S(R)$.

Proof. Denote by H the two-sided ideal which contains $S^*(R)$ and which satisfies $H/S^* = N(R/S^*)$ (= sum of all the nilpotent right-ideals in R/S^*). That H is a two-sided ideal, is a consequence of Lemma 1.1; and it follows from Lemma 1.1 that H is a nilideal modulo S^*. Hence we deduce from Corollary 10.4 that H is part of $S(R)$; and we infer from Theorem 10.2 that the two-sided ideal H is quasi-regular. Hence $H = S^*$ and 0 is the only nilpotent right-ideal in R/S^*. Applying Theorem 10.6 we see that $S^* = S$.

THEOREM 10.9. *If the ring* R *possesses an identity element* 1, *then* $S(R)$ *is part of every maximal right-ideal in* R.

Proof. If $S(R)$ were not contained in the maximal right-ideal G in R, then $R = G + S$ and there exist elements g and s in G and S respectively such that $1 = g + s$. It is a consequence of Theorem 10.2 that $-s$ is quasi-regular and that therefore $-s + t - st = 0$ for some t in R. Hence $1 + t = (g + s)(1 + t) = g + s + gt + st$ or $1 = g + gt$ so that 1 is an element in the right-ideal $G < R$, a contradiction.

COROLLARY 10.10. *If the ring* R *possesses an identity element* 1, *and if* R/S^* *is a sum of minimal right-ideals, then* $S(R) = S^*(R)$ *is the cross-cut of all the maximal right-ideals in* R.

This is an immediate consequence of Lemma 3.3, Corollary 10.8 and Theorem 10.9.

Remark. The upper radical U of the ring R is by Theorem 1.2 a two-sided nilideal; and hence it is a consequence of Theorem 10.2 and of Corollary 10.4 that $U \leq S^*(R)$. That the ideals U and S^* need not be equal may be seen from the following *example*: R is the ring of all the (formal) power series $\sum_{i=0}^{\infty} c_i t^i$ in one indeterminate t with coefficients c_i from some commutative field. An element in R possesses an inverse in R if, and only if, "its absolute term" $c_0 \neq 0$; and an element in R is quasi-regular if, and only if, $1 + c_0 \neq 0$. This shows that $S^*(R) = S(R) = tR$ whereas $U(R) = 0$.

THEOREM 10.11. *If* J *is a quasi-regular right-ideal in the ring* R, *then* $M_v J^v = 0$ *for every ordinal* v.

Remark. It is a consequence of Corollary 10.4 that Theorem 6.1 is a special case of this Theorem 10.11.

Proof. A. If Z is a minimal right-ideal in R, and if z is any element in Z, then zJ is a right-ideal in R which is part of Z. Thus zJ is either 0 or Z. If zJ were equal to Z, then there would exist an element w in J such that $zw = z$. Since w is an element in the quasi-regular right-ideal J, it follows that $-w$ is quasi-regular; and hence there exists an element v in R such that $w = v - wv$. Consequently

$$z = zw = z(v - wv) = zv - zv = 0,$$

an impossibility. Thus we have shown that $zJ = 0$ for every z in Z; and this shows that $ZJ = 0$ for every minimal right-ideal Z in R. Since the anti-radical $M = M(R)$ is the sum of all the minimal right-ideals in R, it follows that

$$MJ = 0 \text{ for every quasi-regular right-ideal } J \text{ in } R.$$

B. We proceed to prove the theorem by complete induction with regard to v. That the theorem holds true for $v = 1$, has been shown under A.; and thus we may assume that our assertion is valid for every $u < v$.

Case 1. $v = w + 1$ is not a limit-ordinal.

Then M_v/M_w is the anti-radical of R/M_w and $(J + M_w)/M_w$ is a quasi-regular right-ideal in R/M_w. Hence it follows from A. that their product is 0; and from this fact we deduce that $M_vJ \leq M_w$. From the induction-hypothesis we infer $M_wJ^w = 0$; and thus we find that

$$M_vJ^v = M_vJJ^w \leq M_wJ^w = 0.$$

Case 2. v is a limit-ordinal.

If x is any element in M_v, then we deduce from the definition of M_v as the join of the M_u with $u < v$ the existence of an ordinal $d < v$ such that x is an element in M_d. It is a consequence of the definition of J^v that it is part of J^d; and hence it follows from the induction-hypothesis that $xJ^v \leq M_dJ^d = 0$; and thus we have shown that $M_vJ^v = 0$.

11. Rings admitting operators. The ring R is said to admit the elements in the system $\boldsymbol{V}$ as operators, if to every element r in R and to every element $\boldsymbol{v}$ in $\boldsymbol{V}$ there exists a uniquely determined element $r\boldsymbol{v}$ in R meeting the following requirements:

(i) $(r \pm s)\boldsymbol{v} = r\boldsymbol{v} \pm s\boldsymbol{v}$;
(ii) $(rs)\boldsymbol{v} = r(s\boldsymbol{v}) = (r\boldsymbol{v})s$.

It is readily verified that $J\boldsymbol{v}$ is a right-(left-) ideal in R whenever J is a right-(left-) ideal in R; that $r^i = 0$ implies $(r\boldsymbol{v})^i = 0$; and that therefore $J\boldsymbol{v}$ is a nilideal (a nilpotent right-ideal) whenever J is a nilideal (a nilpotent right-ideal). Saying that an ideal J in R is *$\boldsymbol{V}$-admissible*, if $J\boldsymbol{V} \leq J$, it is now easy to prove the following statement:

If the ring R admits the elements in the system $\boldsymbol{V}$ as operators, then the upper and the lower radical and all the ideals M_v in the anti-radical series are $\boldsymbol{V}$-admissible.[44]

If the ring R satisfies condition (4. A), then there exists to every element x in R an element x^* in R such that $x = xx^*$; and this shows that $x\boldsymbol{v} = x(x^*\boldsymbol{v})$ for every x in R. Hence we have proved the following theorem:

If condition (4. A) *is satisfied by the ring R, then every right-ideal in R is $\boldsymbol{V}$-admissible.*

On the basis of this theorem it becomes evident that most of the theorems derived in this paper may be applied to rings admitting operators.

UNIVERSITY OF ILLINOIS,
URBANA, ILL.

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RINGS WITH DUALS.*

By REINHOLD BAER.

1. Introduction. If J is a right-ideal in the ring R, then the set $L(J)$ of all the elements x satisfying $xJ = 0$ is a left-ideal in R; in this way there is defined a monotonically decreasing map of the partially ordered set of all the right-ideals in R into the partially ordered set of all the left-ideals in R. A monotonically decreasing and biunique correspondence between the elements of two partially ordered sets has been termed a *duality*; and M. Hall [1] and T. Nakayama [2] have recently started an investigation of those rings in which the operation $L(J)$ defines a duality. Particularly significant is a result, announced by Nakayama,[3] that, under suitable conditions concerning the ring R, the operation $L(J)$ will constitute a duality whenever there exists a duality of the set of right-ideals in R upon the set of left-ideals in R.

Thus the following generalization of these problems seems to be quite natural: the ring R' is said to be *a right-dual of the ring* R, if there exists a duality of the partially ordered set of all the right-ideals in R upon the partially ordered set of all the right-ideals [4] in R'; we propose to investigate the structure of rings possessing a right-dual.

T. Nakayama [5] has given an example of a commutative ring R possessing an identity-element and a duality of the partially ordered set of its ideals upon itself such that neither the maximum nor the minimum condition is satisfied by the ideals in R. Thus the following theorem seems to be noteworthy: The maximum and the minimum condition are satisfied by the right-ideals in the

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[1] Hall (1); the numbers refer to the bibliography at the end of the paper.

[2] Nakayama (1).

[3] Nakayama (2).

[4] It might seem more natural to say that the ring R' is a right-dual of the ring R, if there exists a duality of the partially ordered set of all the right-ideals in R upon the partially ordered set of all the left-ideals in R'. But as far as the ring R is concerned, these two definitions are equivalent, since there exists to every ring R' an anti-isomorphic ring R'', and since, therefore, the existence of a duality of the set of right-ideals in R upon the set of left-ideals in R' implies the existence of a duality of the set of right-ideals in R upon the set of right-ideals in R''. The definition adopted in the text has the great advantage that it concerns itself with right-ideals only, a fact that makes for greater simplicity in notation.

[5] Nakayama (2), p. 55.

ring R, if R possesses a right-identity element, and if there exists to every quotient ring of R a right-dual possessing a left-identity element [Theorem 2.2]. These rings may furthermore be shown to meet a generalized uniseriality requirement [Theorem 3.2].[6]

If the right-ideals in R form a finite ordered set, then R is its own right-dual; there exist[7] examples of such rings where the left ideals do not form a finite ordered set so that the operation $L(J)$ certainly is not a duality. If R is a primary ring[8] with identity element all of whose quotient rings possess right-duals with identity elements, and if the right-ideals in R do not form a finite ordered set, then we are able to prove that $L(J)$ is a duality [Theorem 5.4]; and we give a complete theory of the structure and of the dualities of such rings. [Sections **5** and **6**.]

2. Maximum and minimum conditions for right-ideals. The right-ideal J in the ring R is termed a *minimal* right-ideal, if $0 < J$ and if there does not exist a right-ideal V such that $0 < V < J$. Likewise the right-ideal J in R is said to be a *maximal* right-ideal, if $J < R$ and if there does not exist a right-ideal W such that $J < W < R$.

The set S of right-ideals in R is called *independent*, if the cross-cut of J and of the sum of the right-ideals different from J in S is 0 for every J in S.

The element e in R is a *left-identity element* in R, if $ex = x$ for every element x in R. Right-identity elements are defined analogously and elements that are both right- and left-identity elements are called identity elements (in symbols: 1).

LEMMA 2.1. *If the ring R possesses a right-dual which contains a left-identity element, then R has the following properties:*

(a) *If J_i for $i = 1, 2, \cdots$ are right-ideals such that $0 < J_i < J_{i-1}$ then their cross-cut is different from 0.*

(b) *Every right-ideal not 0 in R contains a minimal right-ideal.*

(c) *There does not exist an infinite independent set of right-ideals in R.*

Proof. Suppose that $\boldsymbol{d}$ is a duality of the system of right-ideals in the ring R upon the system of right-ideals in the ring R' and that R' contains a left-identity element e. If the right-ideals J_i in R satisfy $0 < J_i < J_{i-1}$, and if J is the cross-cut of the right-ideals J_i, then the right-ideals $J_i^{\boldsymbol{d}}$ in R'

[6] This is actually a generalization of the concept of generalized uniserial ring, as defined by Nakayama (1), p. 19, and our concept differs from the one introduced by Nakayama mainly in this that it does not impose any requirements on the left-ideals.

[7] Baer (1).

[8] I. e. a ring which is simple modulo its radical.

satisfy: $J_{i-1}{}^d < J_i{}^d < R'$; and the system of all the elements contained in at least one $J_i{}^d$ is precisely the right-ideal J^d. Since none of the $J_i{}^d$ contains e (as a consequence of $R' = eR'$), neither does J^d contain e. Hence $J^d < R'$ and therefore $0 < J$, showing the validity of (a).

If V is a right-ideal different from 0 in R, then $V^d < R'$. Hence there exists a greatest right-ideal G in R' which contains V^d, but does not contain e. Since $R' = eR'$, it follows that G is a maximal right-ideal in R'. The right-ideal G^{d-1} in R is consequently a minimal right-ideal in R and it is clearly contained in J, i. e. (b) is satisfied by R.

If an infinite independent set of right-ideals in R existed, then there would exist a countably infinite, independent set of right-ideals $J_i \neq 0$ in R. But then the right-ideals $T_i = \sum_{i<j} J_j$ would form a properly descending chain of right-ideals whose cross-cut were 0, contradicting property (a). This completes the proof.

THEOREM 2.2. *If every element x in the ring R is contained in the right-ideal xR, and if there exists to every quotient-ring of R a right-dual which contains a left-identity element, then both the maximum and the minimum condition are satisfied by the right-ideals in the ring R; and R contains a right-identity element.*

Proof. It is immediately evident that conditions (b) and (c) of Lemma 2.1 are satisfied by every quotient ring of R. If the two-sided ideals T_i in R would form a properly descending chain ($T_i < T_{i-1}$), then condition (a) would not be satisfied by the right-ideals in the ring R/T for T the cross-cut of the two-sided ideals T_i; and hence there would not exist properly descending chains of two-sided ideals in R. Now Theorem 2.2 is an immediate consequence of two theorems which we have proved elsewhere.[9]

3. Uniseriality. If the minimum condition is satisfied by the right-ideals in the ring R, then there exists[10] the *radical* $P = P(R)$ of R. This ideal is characterized by the following properties: (1) P is a two-sided ideal; (2) P is a nilpotent ideal, i. e. there exists an integer i such that $P^i = 0$; (3) P contains every nilideal, i. e. every right-ideal J such that there exists to every element x in J an integer n satisfying: $x^n = 0$. If there exists a left-identity element in R, then it can be shown[11] that (4) P is the cross-cut

[9] Baer (3), Theorems 8.4 and 9.4.

[10] Hopkins (1), p. 714, Theorem 1.4.

[11] Baer (3), Theorem 3.4. If one considers Baer (3), Theorem 9.6, then one sees that this condition (4) and the existence of a left-identity element are almost equivalent requirements.

of all the maximal right-ideals in R. A right-ideal in R is termed *regular*, if it is not part of the radical P of R. If the minimum condition is satisfied by the right-ideals in R, then we may deduce from a theorem due to Köthe [12] that every regular right-ideal contains an idempotent not 0. This implies in particular that

minimal regular right-ideals are direct summands of R;

here we term a regular right-ideal minimal, if no proper subideal is regular.

Consideration of property (4) of the radical of R leads to the definition of the *anti-radical* $M = M(R)$ of the ring R as the sum of all the minimal right-ideals in R. It is a two-sided ideal in R; if the right-ideal J is part of M, then J is a direct summand of M and J is the direct sum of minimal right-ideals.[13]

LEMMA 3.1. *Suppose that the minimum condition is satisfied by the right-ideals in the ring R, that R contains a left-identity element and that there exists a right-dual of R which possesses a left-identity element. Then the regular right-ideal J in R is a minimal regular right-ideal if,*[14] *and only if, J contains one and only one minimal right-ideal in R.*

Proof. Assume first that the regular right-ideal J in R contains one and only one minimal right-ideal in R. Since the minimum condition is satisfied by the right-ideals in R, there exists a minimal regular right-ideal J^* which is part of J. It has been pointed out before that J^* is a direct summand of the ring R, and hence there exists a right-ideal J^{**} such that J is the direct sum of J^* and J^{**}. Since J^* is regular, it contains a minimal right-ideal in R. Since J contains only one minimal right-ideal, it follows that J^{**} does not contain any minimal right-ideal; and consequently we may deduce from the minimum condition that $J^{**} = 0$, i. e. that $J = J^*$ is a minimal regular right-ideal.

Assume now, conversely, that J is a minimal regular right-ideal in R. Then there exists a right-ideal V in R such that R is the direct sum of J and V. There exist furthermore a ring R' which contains a left-identity element and a duality $\boldsymbol{d}$ of the system of all the right-ideals in R upon the system of all the right-ideals in R'. It is a consequence of a theorem of Ch. Hopkins [15] that the maximum condition is satisfied by the right-ideals in R;

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[12] Köthe (1), p. 168, Satz 7.

[13] Cp. e. g. Baer (3), Section 4, or Hopkins (1).

[14] It should be noted that the existence of a dual is not needed for proving the sufficiency of this condition.

[15] Hopkins (1), p. 726, Theorem 6.4.

and thus it follows that both the maximum and the minimum condition are satisfied by the right-ideals in R'. Using the property (4) of the radical, mentioned before, we see now that $P(R)^{\boldsymbol{d}} = M(R')$, $M(R)^{\boldsymbol{d}} = P(R')$. It is readily seen that $(P(R) + J)/P(R)$ is a minimal right-ideal in the ring $R/P(R)$ and that consequently $P(R) + V$ is a maximal right-ideal in R. The duality $\boldsymbol{d}$ maps therefore $P(R) + V$ upon a minimal right-ideal in R'; and since $(P(R) + V)^{\boldsymbol{d}}$ is just the cross-cut of $M(R')$ and $V^{\boldsymbol{d}}$, it follows that $V^{\boldsymbol{d}}$ contains one and only one minimal right-ideal in R'. Hence it follows from the first part of this lemma that $V^{\boldsymbol{d}}$ is a minimal regular right-ideal in R'; and R' is the direct sum of $J^{\boldsymbol{d}}$ and $V^{\boldsymbol{d}}$, since R is the direct sum of J and V. The arguments used in proving that $V^{\boldsymbol{d}}$ contains one and only one minimal right-ideal may now be applied on $J^{\boldsymbol{d}}$ to show that J contains one and only one minimal right-ideal. This completes the proof of the lemma.

The element c in the partially ordered set S is said to be a *cycle of order* n, if there exist exactly $n + 1$ elements x in S which satisfy: $x \leqq c$, and if the parts of c in S form an ordered set. Thus the minimal right-ideals in the ring R are just the cycles of order 1 in the partially ordered set of all the right-ideals in R.

The two-sided ideal $M_i = M_i(R)$ in the ring R is defined inductively by the following rules: $M_0 = 0$, $M_{i+1}/M_i = M(R/M_i)$. If the minimum-condition is satisfied by the right-ideals in R, then $M_i < R$ implies $M_i < M_{i+1}$; and if both the maximum and the minimum condition are satisfied by the right-ideals in R, then there exists a positive integer m such that $R = M_m$.

THEOREM 3.2. *Suppose that the minimum condition is satisfied by the right-ideals in the ring R, that R contains a left-identity element, and that there exists to every quotient-ring of R a right-dual which possesses a left-identity element. If the minimal regular right-ideal J in R is contained in M_i, but is not contained in M_{i-1}, then J is a cycle of order i in the partially ordered set of right-ideals in R.*

Remarks. 1. The hypothesis that the minimum condition be satisfied by the right-ideals in R cannot be deduced from the other hypotheses by means of Theorem 2.2, since we had to impose there a hypothesis which amounted to the existence of a right-identity element, whereas here we assume the existence of a left-identity element.

2. It is a consequence of a theorem of Ch. Hopkins [15] that the maximum condition is satisfied by the right-ideals in the ring R. Hence there exists, as has been remarked before, a positive integer m such that $R = M_m$; and

of all the maximal right-ideals in R. A right-ideal in R is termed *regular*, if it is not part of the radical P of R. If the minimum condition is satisfied by the right-ideals in R, then we may deduce from a theorem due to Köthe [12] that every regular right-ideal contains an idempotent not 0. This implies in particular that

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Assume now, conversely, that J is a minimal regular right-ideal in R. Then there exists a right-ideal V in R such that R is the direct sum of J and V. There exist furthermore a ring R' which contains a left-identity element and a duality $\boldsymbol{d}$ of the system of all the right-ideals in R upon the system of all the right-ideals in R'. It is a consequence of a theorem of Ch. Hopkins [15] that the maximum condition is satisfied by the right-ideals in R;

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[12] Köthe (1), p. 168, Satz 7.

[13] Cp. e. g. Baer (3), Section 4, or Hopkins (1).

[14] It should be noted that the existence of a dual is not needed for proving the sufficiency of this condition.

[15] Hopkins (1), p. 726, Theorem 6.4.

and thus it follows that both the maximum and the minimum condition are satisfied by the right-ideals in R'. Using the property (4) of the radical, mentioned before, we see now that $P(R)^{\mathfrak{d}} = M(R')$, $M(R)^{\mathfrak{d}} = P(R')$. It is readily seen that $(P(R) + J)/P(R)$ is a minimal right-ideal in the ring $R/P(R)$ and that consequently $P(R) + V$ is a maximal right-ideal in R. The duality $\mathfrak{d}$ maps therefore $P(R) + V$ upon a minimal right-ideal in R'; and since $(P(R) + V)^{\mathfrak{d}}$ is just the cross-cut of $M(R')$ and $V^{\mathfrak{d}}$, it follows that $V^{\mathfrak{d}}$ contains one and only one minimal right-ideal in R'. Hence it follows from the first part of this lemma that $V^{\mathfrak{d}}$ is a minimal regular right-ideal in R'; and R' is the direct sum of $J^{\mathfrak{d}}$ and $V^{\mathfrak{d}}$, since R is the direct sum of J and V. The arguments used in proving that $V^{\mathfrak{d}}$ contains one and only one minimal right-ideal may now be applied on $J^{\mathfrak{d}}$ to show that J contains one and only one minimal right-ideal. This completes the proof of the lemma.

The element c in the partially ordered set S is said to be a *cycle of order* n, if there exist exactly $n + 1$ elements x in S which satisfy: $x \leqq c$, and if the parts of c in S form an ordered set. Thus the minimal right-ideals in the ring R are just the cycles of order 1 in the partially ordered set of all the right-ideals in R.

The two-sided ideal $M_i = M_i(R)$ in the ring R is defined inductively by the following rules: $M_0 = 0$, $M_{i+1}/M_i = M(R/M_i)$. If the minimum-condition is satisfied by the right-ideals in R, then $M_i < R$ implies $M_i < M_{i+1}$; and if both the maximum and the minimum condition are satisfied by the right-ideals in R, then there exists a positive integer m such that $R = M_m$.

THEOREM 3.2. *Suppose that the minimum condition is satisfied by the right-ideals in the ring R, that R contains a left-identity element, and that there exists to every quotient-ring of R a right-dual which possesses a left-identity element. If the minimal regular right-ideal J in R is contained in M_i, but is not contained in M_{i-1}, then J is a cycle of order i in the partially ordered set of right-ideals in R.*

Remarks. 1. The hypothesis that the minimum condition be satisfied by the right-ideals in R cannot be deduced from the other hypotheses by means of Theorem 2.2, since we had to impose there a hypothesis which amounted to the existence of a right-identity element, whereas here we assume the existence of a left-identity element.

2. It is a consequence of a theorem of Ch. Hopkins [15] that the maximum condition is satisfied by the right-ideals in the ring R. Hence there exists, as has been remarked before, a positive integer m such that $R = M_m$; and

consequently there exists to every right-ideal J in R an integer i such that $J \leq M_i$, $J \neq M_{i-1}$.

Proof. If $0 < j \leq i$, then $(J + M_{i-j})/M_{i-j}$ is a minimal regular right-ideal in the quotient-ring R/M_{i-j} which is contained in $M_j(R/M_{i-j})$, but not in $M_{j-1}(R/M_{i-j})$. We proceed to prove by complete induction with regard to j that $(J + M_{i-j})/M_{i-j}$ is a cycle of order j in the partially ordered set of all the right-ideals in R/M_{i-j}. It follows from $M = M_1$ and from Lemma 3.1 that $(J + M_{i-1})/M_{i-1}$ is a minimal right-ideal in R/M_{i-1}, i. e. a cycle of order 1 in the partially ordered set of all the right-ideals in R/M_{i-1}. Thus we may assume that our contention holds true for $j - 1$, i. e. that $(J + M_{i-j+1})/M_{i-j+1}$ is a cycle of order $j - 1$ in the partially ordered set of all the right-ideals in R/M_{i-j+1}. We note that $M_{i-j+1}/M_{i-j} = M(R/M_{i-j})$; and hence we may infer from Lemma 3.1 that the cross-cut of $(J + M_{i-j})/M_{i-j}$ and M_{i-j+1}/M_{i-j} is a cycle of order 1 (= a minimal right-ideal in R/M_{i-j}); and now it is readily seen that $(J + M_{i-j})/M_{i-j}$ is a cycle of order j in the partially ordered set of all the right-ideals in R/M_{i-j}. This completes the inductive proof. Hence it follows, in particular, that J is a cycle of order i in the partially ordered set of all the right-ideals in R, since $M_0 = 0$.

4. Duals of primary abelian operator groups. The abelian group G admitting the elements in the ring E as operators is said to be a *primary abelian operator group over E*, if

(i) E contains an identity element 1 satisfying $g1 = g$ for every g in G and $x1 = 1x = x$ for every x in E, and if

(ii) E is a cycle of order $m = m(E)$ in the partially ordered set of all the right-ideals in E.

We note that E is a (not necessarily commutative) field if, and only if, $m(E) = 1$; and one may verify readily that a ring E satisfying condition (ii) has the following properties: each right-ideal in E is a two-sided ideal and is furthermore a power of the radical $P = P(E)$.

If g is an element in the primary abelian operator group G over E, then gE is the E-admissible subgroup of G generated by g; and it is a consequence of condition (ii) that gE is a cycle of an order not exceeding $m(E)$ in the partially ordered set of all the E-admissible subgroups of G. We note finally that all the admissible subgroups and quotient groups of primary abelian operator groups over E are themselves primary abelian operator groups over E.

THEOREM 4.1. *If the primary abelian operator group G over E contains*

at least two independent cycles of order $m(E)$, and if there exists a primary abelian operator group G' over E' such that the partially ordered set of all the E'-admissible subgroups of G' is the dual of the partially ordered set of all the E-admissible subgroups of G, then every left-ideal in E is a two-sided ideal in E.

Proof. Denote by $\boldsymbol{d}$ a duality of the set of all the E-admissible subgroups of G upon the set of all the E'-admissible subgroups of G'. Suppose furthermore that U and V are E-admissible subgroups of G such that $U < V$ and such that there exist at most two different cycles of order 1 in the partially ordered set of the E-admissible subgroups of V/U. It is easily seen that there exists one and only one E-admissible subgroup W of G such that $U < W \leq V$ and such that W/U is a cycle of order 1 in the partially ordered set of all the E-admissible subgroups of V/U. Since $W^{\boldsymbol{d}} < U^{\boldsymbol{d}}$, there exists an element z in $U^{\boldsymbol{d}}$ which is not contained in $W^{\boldsymbol{d}}$. There exists one and only one E-admissible subgroup Z of G such that $Z^{\boldsymbol{d}} = V^{\boldsymbol{d}} + zE'$. Then $V^{\boldsymbol{d}} \leq Z^{\boldsymbol{d}} \leq U^{\boldsymbol{d}}$, but $Z^{\boldsymbol{d}} \not\leq W^{\boldsymbol{d}}$; and hence $U \leq Z \leq V$, but $W \not\leq Z$. Since W/U is the only cycle of order 1 in V/U, it is part of every subgroup not 0 of V/U; and thus it follows that $Z = U$ or $U^{\boldsymbol{d}} = V^{\boldsymbol{d}} + zE'$, showing that $U^{\boldsymbol{d}}/V^{\boldsymbol{d}}$ is a cycle. But the dual of a cycle is a cycle, i. e. V/U is a cycle in the partially ordered set of the E-admissible subgroups of G/U. But it has been shown elsewhere [16] that this condition and the hypotheses of the Theorem assure the two-sidedness of the left-ideals in E.

5. Primary rings with duals. A ring R is said to be *primary*, if it meets the following requirements:

(a) R contains an identity element 1.

(b) The minimum condition is satisfied by the right-ideals in R.

(c) There does not exist a two-sided ideal T in R satisfying: $P < T < R$ (where $P = P(R)$ is the radical of R).

A ring R is said to be *completely primary*, if it satisfies the conditions (a), (b) and has the following stronger property:

(c*) There does not exist a right-ideal J in R satisfying: $P < J < R$.

The condition (c*) may be seen to be equivalent to the more symmetric condition:

(c**) R/P is a (not necessarily commutative) field.

It is a consequence of a theorem of Ch. Hopkins [17] that the maximum

[16] Baer (1).

[17] Hopkins (1), p. 726, Theorem 6. 4.

condition is satisfied by the right-ideals in a primary ring R; and thus the following facts are restatements of results of G. Köthe:[18]

LEMMA 5.1. *The ring R is primary if, and only if, R is the ring E_n of all the n by n matrices with coefficients from the completely primary ring E. The order n and the completely primary ring E are [essentially] uniquely determined by the primary ring $R = E_n$.*

A ring is termed *uni-serial*, if every minimal regular right-ideal is a cycle in the partially ordered set of all its right-ideals.[19] The following statement may be proved by arguments which K. Asano[20] used for the proof of a similar fact:

LEMMA 5.2. *A ring E meeting the requirements* (a) *and* (b) *above is a completely primary, uni-serial ring if, and only if, every right-ideal in E has the form $P(E)^i$ for suitable i; and the completely primary ring E is uni-serial if, and only if, E_n is uni-serial.*

We are now ready to prove that primarity and uni-seriality are invariant under dualities.

THEOREM 5.3. *Suppose that the minimum condition is satisfied by the right-ideals in the rings R and R', that each of these rings contains an identity element and that there exists a duality $\mathbf{d}$ of the partially ordered set of all the right-ideals in R upon the partially ordered set of all the right-ideals in R'.*

(i) *If R is uni-serial, then R' is uni-serial.*

(ii) *If R is primary, then R' is primary.*

Proof. Suppose first that R is uni-serial and that J' is a minimal regular right-ideal in R'. Then J' is, as has been pointed out in section **3**, a direct summand of R', i.e. R' is the direct sum of J' and of some right-ideal J''. It is a consequence of Lemma 3.1 that J' contains one and only one minimal right-ideal; and J'' is therefore a greatest right-ideal not containing $M(R')$. Using the property (4) of the radical, enunciated at the beginning of section **3**, it follows that $M(R')^{\mathbf{d}^{-1}} = P(R)$ and $M(R)^{\mathbf{d}} = P(R')$. Hence $J''^{\mathbf{d}^{-1}}$ is a minimal regular right-ideal in R and is therefore a cycle in the partially ordered set of all the right-ideals in R. But R is the direct sum of $J'^{\mathbf{d}^{-1}}$ and $J''^{\mathbf{d}^{-1}}$, since R' is the direct sum of J' and J''. Thus R is a cycle in the partially

[18] Köthe (1), p. 176, Hauptsatz 12 and p. 179, Zusatz 2.

[19] Note that this definition of uni-seriality does not impose any requirements on the left-ideals; cf. [6].

[20] Asano (1), p. 233/234, Hilfssätze 3, 4.

ordered set of all the right-ideals in R which contain J'^{d-1}; and J' is consequently a cycle in the partially ordered set of the right-ideals in R', i. e. R' is uni-serial.

Suppose now that R is primary and that T' is a two-sided ideal in R' such that $P(R') < T' < R'$. Since $R'/P(R')$ is a semi-simple ring, there exists [21] a two-sided ideal T'' such that $R' = T' + T''$ and such that $P(R')$ is the cross-cut of T' and T''. If $S' = T'^{d-1}$ and S''^{d-1}, then $M(R)$ is the direct sum of S' and S'', since the sum of T' and T'' is R', and since the cross-cut of T' and T'' is just $P(R') = M(R)^d$. Every minimal right-ideal of $R'/P(R')$ is contained in one and only one of the two-sided ideals $T'/P(R')$ and $T''/P(R')$ whose direct sum is the ring $R'/P(R')$; and this implies that every greatest proper subideal of $M(R)$ contains one and only one of the two ideals S' and S''.

It is a consequence of Lemma 5.1 that R is the ring E_n of all the n by n matrices with coefficients from the completely primary ring E. The ring E is isomorphic to a ring eRe where e is a primitive idempotent in R (i. e. an idempotent different from 0 which is not the sum of two orthogonal idempotents both of which are different from 0); and it is a consequence of the fundamental property of minimal regular right-ideals (mentioned in section **3**) that eR is a minimal regular right-ideal. Hence we deduce from Lemma 3.1 that eR contains one and only one minimal right-ideal; and this makes it evident that E itself contains one and only one minimal right-ideal, or $M(E)$ is a minimal right-ideal in E.

If z is any element not 0 in $M(E)$, then $M(E) = zE$, since E is a minimal right-ideal in E. Since $P(E)$ is nilpotent, $M(E)P(E) = 0$. The set Z of all the elements x in E which satisfy: $zx = 0$ is a right-ideal satisfying: $P(E) \leq Z < E$; and since E is a completely primary ring, it follows that $Z = P(E)$.

We note next the facts that $M(R) = M(E_n)$ is the set of all the matrices in E_n whose coefficients are in $M(E)$; and that $P(R) = P(E_n)$ is the set of all the matrices in E_n whose coefficients are in $P(E)$.

Let z be any element not 0 in $M(E)$; and define, for every matrix (a_{ij}) in E_n, the map $\boldsymbol{f}$ by $(a_{ij})^{\boldsymbol{f}} = (za_{ij})$. Clearly $\boldsymbol{f}$ is a single-valued transformation which preserves addition and which satisfies: $R^{\boldsymbol{f}} = M(R)$, since $zE = M(E)$. It has been shown that $zx = 0$ if, and only if, x is in $P(E)$. Consequently $(a_{ij})^{\boldsymbol{f}} = 0$ if, and only if, every a_{ij} is in $P(E)$ i. e. if, and only if, (a_{ij}) is in $P(R)$. This shows that $\boldsymbol{f}$ induces an isomorphism of the

[21] Cf. e. g. v. d. Waerden (1), p. 170.

additive group of the ring $R^* = R/P(R)$ upon $M(R)$ and that this induced isomorphism maps right-ideals in R^* upon the right-ideals in R which are part of $M(R)$. If V' and V'' are the right-ideals in R^* which are mapped by $\mathfrak{f}$ upon the right-ideals S' and S'' respectively, then it follows from the properties of S' and S'' that the ring R^* is the direct sum of the right-ideals V' and V''; and every maximal right-ideal in R^* contains one and only one of the ideals V' and V''.

The ring R^* is by its construction essentially the same as the ring of all n by n matrices with coefficients from the field $E^* = E/P(E)$. Thus there exists a system e_{ij} of n^2 matrix units in R^* satisfying: $e_{ij}e_{hk} = \begin{cases} 0 \text{ for } j \neq h \\ e_{ik} \text{ for } j = h \end{cases}$, and the elements e_{ii} are a set of n mutually orthogonal primitive idempotents in R^*.

The right-ideal $\sum_{j \neq i} e_{jj}R^*$ is clearly a maximal right-ideal in R^* so that it contains one and only one of the ideals V' and V''. We may assume, therefore, without loss in generality, that

$$V'' \leq \sum_{j \neq i} e_{jj}R^* \text{ for } i = 1, \cdots, h; \quad V' \leq \sum_{j \neq i} e_{jj}R^* \text{ for } i = h+1, \cdots, n;$$

and we deduce from the fact that R^* is the direct sum of V' and V'' as well as the direct sum of the $e_{jj}R^*$ that

$$V' = \sum_{j=1}^{h} e_{jj}R^*, \qquad V'' = \sum_{j=h+1}^{n} e_{jj}R^*.$$

It is readily verified that the $n-1$ elements $e_{11} + e_{n1}, e_{22}, \cdots, e_{n-1,n-1}$ are mutually orthogonal idempotents of which the first is different from 0 whereas the $n-2$ others are primitive idempotents. If e_{nn} were an element in the right-ideal $W = (e_{11} + e_{n1})R^* + \sum_{i=2}^{n-1} e_{ii}R^*$, then $e_{nn} = (e_{11} + e_{n1})r_1 + \sum_{i=2}^{n-1} e_{ii}r_i$. If $0 < i < n$, then $0 = e_{ii}e_{nn} = e_{ii}r_i$ so that $e_{nn} = (e_{11} + e_{n1})r_1 = e_{n1}r_1$ and hence $e_{1n} = e_{1n}e_{nn} = e_{1n}e_{n1}r_1 = e_{11}r_1 = 0$, an impossibility. Thus $W < R^*$, showing that W is a maximal right-ideal in R^* which does not contain e_{nn}. Thus in particular V'' is not part of W. Since one of the ideals V' and V'' is contained in W, we have found that $V' \leq W$. Consequently e_{11} is an element in W and therefore e_{n1} and $e_{nn} = e_{n1}e_{1n}$ are elements in W. This impossibility shows that the original hypothesis of non-primarity of R' has led us to a contradiction; and this completes the proof of the theorem.

THEOREM 5.4. *Suppose that the ring E is completely primary and uni-serial. Then*

(I) E_1 $(= E)$ *is its own right-dual.*

(II) *There exists a right-dual to* E_n *for* $n > 1$ *which contains an identity element if, and only if, every left-ideal in* E *is a right-ideal.*

Proof. The assertion (I) is an obvious consequence of the fact that the right-ideals in E form a finite ordered set.

If every left-ideal in E is a right-ideal, then every ideal in E (left-ideals as well as right-ideals) is a power of the radical $P(E)$ and it is known [22] that a duality of the partially ordered set of all the right-ideals in E_n upon the partially ordered set of all the left-ideals in E_n is effected by mapping the right-ideal J in E_n upon the set $L(J)$ of all the elements x in E_n which satisfy $xJ = 0$.

Suppose finally that $1 < n$ and that there exists a duality $\boldsymbol{d}$ of the partially ordered set of all the right-ideals in E_n upon the partially ordered set of all the right-ideals in the ring R' which contains an identity element. It is a consequence of Lemmas 5.1 and 5.2 that the ring E_n is primary and uni-serial; and thus we deduce from Theorem 5.3 that R' is primary and uni-serial. There exist by Lemmas 5.1 and 5.2 an integer k and a completely primary, uni-serial ring F such that $R' = F_k$.

Denote by (E, n) the abelian group of all the n-tuplets of elements in E; and by (F, k) the abelian group of all the k-tuplets of elements in F. The group (E, n) admits the elements in E as operators (multiplication from the right), is a primary abelian operator group over E (in the sense of section **4**) and is the direct sum of n cycles of order $m(E)$; similar statements may be made concerning (F, k).

If B is an E-admissible subgroup of (E, n), then denote by $B^{\boldsymbol{p}}$ the set of all those matrices in E_n all of whose columns are n-tuplets in B; and if J is a right-ideal in F_k, then denote by $J^{\boldsymbol{q}}$ the set of all the k-tuplets in (F, k) which are columns of matrices in J. It is readily verified that $\boldsymbol{p}$ is a projectivity [23] of the partially ordered set of all the E-admissible subgroups of (E, n) upon the partially ordered set of all the right-ideals in E_n; and that $\boldsymbol{q}$ is a projectivity of the partially ordered set of all the right-ideals in F_k upon the partially ordered set of all the F-admissible subgroups of (F, k). The transformation $\boldsymbol{pdq}$ is therefore a duality of the partially ordered set of all the E-admissible subgroups of (E, n) upon the partially ordered set of all the F-admissible subgroups of (F, k); and hence it follows from Theorem 4.1 that every left-ideal in E is a two-sided ideal in E, as was to be shown.

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[22] Cf. e.g. Baer (2), Theorem 7.5, p. 210 or Nakayama (1), Theorem 6, p. 9.

[23] A projectivity between two partially ordered sets is a biunique and monotonically increasing correspondence between the elements of these partially ordered sets.

It will be convenient to term a ring R *quasi-primary*, if R contains an identity-element 1, and if there exists in R a two-sided ideal $T < R$ such that every two-sided ideal different from R in R is part of T. That every primary ring is quasi-primary, is a consequence of the following slightly more general statement.

LEMMA 5.5. *If the ring R contains an identity element* 1, *and if there exists in R a two-sided ideal T, all of whose elements are nil-elements,*[24] *such that R/T is a simple ring,*[25] *then the ring R is quasi-primary.*

Proof. Suppose that the two-sided ideal S in R is not part of T. Then $T < S + T$ and $S + T$ is a two-sided ideal so that $R = S + T$. Hence there exist elements s and t in S and T, respectively, such that $1 = s + t$. Consequently $1 = (s + t)^n = s_n + t^n$ for every n and s_n an element in the two-sided ideal S. Since t is a nil-element, $t^i = 0$ for some i. Thus 1 is in S and $S = R$, as was to be shown.

THEOREM 5.6. *The ring R is quasi-primary and there exists to every quotient-ring of R a right-dual possessing an identity element* 1 *if, and only if, R satisfies the following two conditions*:

(i) *R is the ring E_n of all the n by n matrices with coefficients from the completely primary and uni-serial ring E.*

(ii) *Either $n = 1$ or else every left-ideal in E is a right-ideal in E.*

Proof. If R is a quasi-primary ring whose quotient-rings possess duals with identity elements, then we deduce from Theorems 2.2 and 3.2 that the minimum condition is satisfied by the right-ideals in R and that R is uni-serial. Consequently the radical $P(R)$ of R exists. Since R is quasi-primary, there exists one and only one greatest two-sided ideal T different from R. Clearly $P(R) \leq T$. Since $R/P(R)$ is semi-simple, there exists a two-sided ideal T' such that $R = T + T'$ and such that $P(R)$ is the cross-cut of T and T'. It is impossible that T' is part of T and hence we have $R = T'$. Thus the cross-cut of R and T is the radical $P(R)$ of R showing that $R/P(R)$ is simple, i. e. R is a primary ring. It is a consequence of Lemmas 5.1 and 5.2 that there exist a positive integer n and a completely primary, uni-serial ring E such that $R = E_n$; and the necessity of conditions (i) and (ii) is now a consequence of Theorem 5.4.

If the ring R satisfies conditions (i) and (ii), then we infer from Lemma

[24] The element x is a nilelement, if $x^i = 0$ for some i.

[25] I. e. if there does not exist a two-sided ideal T' such that $T < T' < R$.

5.5 that R is quasi-primary. It is easily seen that every quotient-ring of R meets the requirements (i) and (ii); and the sufficiency of (i) and (ii) is an immediate consequence of Theorem 5.4.

We note that we have proved the following fact:

If the maximum condition is satisfied by the right-ideals in the quasi-primary ring R, then R is a primary ring.

6. The dualities of primary, uni-serial rings. If S is a subset of the ring R, then we denote by $L(S)$ the set of all the elements x in R which satisfy: $xS = 0$; and we denote by $K(S)$ the set of all the elements y in R such that $Sy = 0$. Clearly $L(S)$ is a left-ideal and $K(S)$ a right-ideal in R; and both these functions are monotonically decreasing.

If E is a completely primary and uni-serial ring all of whose left-ideals are two-sided ideals, then the following statements hold: [26]

(1) $K(L(J)) = J$ for every right-ideal J in E.

(2) $L(K(J)) = J$ for every left-ideal J in E_n.

(3) The functions K and L define dualities between the partially ordered set of all the right-ideals in E_n and the partially ordered set of all the left-ideals in E_n.

THEOREM 6.1. *If the ring E is uni-serial and completely primary, if $2 < n$, and if $\boldsymbol{d}$ is a duality of the partially ordered set of all the right-ideals in E_n upon the partially ordered set of all the left-ideals in the ring R with identity element* 1, *then there exists an isomorphism $\boldsymbol{f}$ of the ring E_n upon the ring R such that $J^d = L(J^f)$ for every right-ideal J in E_n.*

Proof. It is a consequence of Theorem 5.4, (II), of $2 < n$, and of the existence of the duality $\boldsymbol{d}$ that every left-ideal in E is a two-sided ideal in E. It is a consequence of Lemmas 5.1 and 5.2 that the ring E_n is primary and uni-serial; and hence we deduce from Theorem 5.3 that the ring R is primary and that every minimal regular left-ideal in R is a cycle in the partially ordered set of all the left-ideals in R, i.e. R is left-sided uni-serial. Now we infer from Lemmas 5.1 and 5.2 and from Theorem 5.4, (II) the existence of a completely primary and uni-serial ring F, all of whose left-ideals are two-sided, such that $R = F_k$ for some positive integer k (note that certainly $1 < k$). It is a consequence of the remark (3) above that a projectivity of the set of all the right-ideals in E_n upon the set of all the right-ideals in $R = F_k$ is obtained by mapping the right-ideal J in E_n upon the right-ideal $K(J^d)$ in F_k.

[26] Cf. footnote [22] above.

Denote as before by (E, n) the primary abelian operator group over E which consists of all the n-tuplets of elements in E, and where the elements in E operate upon the group (E, n) by multiplication from the right. If $a = (a_1, \cdots, a_n)$ is an element in (E, n), and if $b = (b_{ij})$ is an element in E_n, then $ba = (\cdots, \sum_{i=1}^{n} b_{ji}a_i, \cdots)$ is a well-determined element in (E, n). If J is a right-ideal in E_n, then $J(E, n)$ is an E-admissible subgroup of (E, n) which consists exactly of all the columns of matrices in J; and it is readily verified that a projectivity of the partially ordered set of all the right-ideals in E_n upon the partially ordered set of all the E-admissible subgroups of (E, n) is obtained by mapping the right-ideal J in E_n upon $J(E, n)$. Analogous statements may be made concerning F_k and (F, k).

If J is a right-ideal in E_n, then we put $(J(E, n))^{\mathfrak{p}} = K(J^{\mathfrak{d}})(F, k)$; and it follows, from the results stated at the end of the first and second paragraph of the proof, that $\mathfrak{p}$ is a projectivity of the partially ordered set of all the E-admissible subgroups of (E, n) upon the partially ordered set of all the F-admissible subgroups of (F, k). Since $2 < n$, we may apply a fundamental theorem on the projectivities of primary abelian operator groups;[27] and hence there exists an isomorphism $\mathfrak{g}$ of E upon F, an isomorphism $\mathfrak{g}$ of (E, n) upon (F, k) such that $(ae)^{\mathfrak{g}} = a^{\mathfrak{g}}e^{\mathfrak{g}}$ for a in (E, n) and e in E and such that $S^{\mathfrak{g}} = S^{\mathfrak{p}}$ for every E-admissible subgroup S of (E, n). There exists an isomorphism $\mathfrak{f}$ of E_n upon F_k such that $(ba)^{\mathfrak{g}} = b^{\mathfrak{f}}a^{\mathfrak{g}}$ for every b in E_n and every a in (E, n), since E_n is anti-isomorphic to the ring of linear transformations of (E, n) and F_k is anti-isomorphic to the ring of linear transformations of (F, k). Now we find that

$$K(J^{\mathfrak{d}})(F, k) = (J(E, n))^{\mathfrak{p}} = (J(E, n))^{\mathfrak{g}} = J^{\mathfrak{f}}(F, k) \quad \text{or} \quad K(J^{\mathfrak{d}}) = J^{\mathfrak{f}};$$

and we infer from (1) above that $L(J^{\mathfrak{f}}) = J^{\mathfrak{d}}$ for every right-ideal J in E_n.

COROLLARY 6.2. *If the ring E is uni-serial and completely primary, if $2 < n$, and if $\mathfrak{d}$ is a duality of the partially ordered set of all the right-ideals in E_n upon the partially ordered set of all the right-ideals in the ring R with identity element 1, then there exists an anti-isomorphism $\mathfrak{f}$ of E_n upon R such that $J^{\mathfrak{d}} = K(J^{\mathfrak{f}})$ for every right-ideal J in E.*

Proof. There exists an anti-isomorphism $\mathfrak{h}$ of the ring R upon a suitable ring R^*. If J is a right-ideal in E_n, then $J^{\mathfrak{d}\mathfrak{h}}$ is a left-ideal in R^*; and $\mathfrak{d}\mathfrak{h}$ is a duality of the partially ordered set of all the right-ideals in E_n upon the partially ordered set of all the left-ideals in R^*. Hence there exists by Theorem 6.1 an isomorphism $\mathfrak{r}$ of the ring E_n upon the ring R^* such that

[27] Baer (1), Theorem II, 3.1, p. 311.

$J^{dh} = L(J^r)$ for every right-ideal J in E_n. It is clear that $\boldsymbol{f} = \boldsymbol{r}\boldsymbol{h}^{-1}$ is an anti-isomorphism of E_n upon R which satisfies:

$$J^d = (J^{dh})^{h^{-1}} = (L(J^r))^{h^{-1}} = K(J^{rh^{-1}}) = K(J^f).$$

Remark. The hypothesis: $2 < n$ cannot be omitted in Theorem 6.1 and Corollary 6.2. For suppose that E and F are two countably infinite, commutative fields which are not isomorphic. Then every proper right-ideal in E_2 or F_2 is a minimal right-ideal; and the same holds for the left-ideals. Thus there exists a duality of the partially ordered set of all the right-ideals in E_2 upon the partially ordered set of all the left-(right-)ideals in F_2, though the rings E_2 and F_2 are neither isomorphic nor anti-isomorphic.

COROLLARY 6.3. *Suppose that E is a completely primary and uni-serial ring, that $2 < n$, and that the ring R contains an identity element* 1. *Then there exists a duality of the partially ordered set of all the right-ideals in E_n upon the partially ordered set of all the left-(right-)ideals in R if, and only if, the rings E_n and R are isomorphic (anti-isomorphic) rings and every left-ideal in E is a right-ideal in E.*

Proof. The necessity of the conditions is an immediate consequence of Theorems 5.4 and 6.1 and of Corollary 6.2. If, conversely, every left-ideal in E is a right-ideal in E, then it follows from the proposition (3), stated at the beginning of this section **6**, that a duality of the partially ordered set of all the right-ideals in E_n upon the partially ordered set of all the left-ideals in E_n is obtained by mapping the right-ideal J in E_n upon the left-ideal $L(J)$ in E_n. If $\boldsymbol{f}$ is an isomorphism of the ring E_n upon the ring $E_n{}^{\boldsymbol{f}}$, then a duality of the set of all the right-ideals in E_n upon the set of all the left-ideals in $E_n{}^{\boldsymbol{f}}$ is effected by mapping the right-ideal J in E_n upon $L(J)^{\boldsymbol{f}}$; and if $\boldsymbol{g}$ is an anti-isomorphism of the ring E_n upon the ring $E_n{}^{\boldsymbol{g}}$, then a duality of the set of all the right-ideals in E_n upon the set of the right-ideals in $E_n{}^{\boldsymbol{g}}$ is obtained by mapping the right-ideal J in E_n upon the right-ideal $L(J)^{\boldsymbol{g}}$ in $E_n{}^{\boldsymbol{g}}$.

COROLLARY 6.4. *Suppose that E is a completely primary and uni-serial ring and that $2 < n$.*

(a) *There exists a duality of the partially ordered set of all the right-ideals in E_n upon the partially ordered set of all the left-ideals in E_n if, and only if, every left-ideal in E is two-sided.*

(b) *There exists a duality of the partially ordered set of all the right-ideals in E_n upon the partially ordered set of all the right-ideals in E_n if, and only if, there exists an anti-automorphism of E.*

Proof. The statement (a) is an immediate consequence of Corollary 6.3, since every ring is isomorphic to itself. If there exists an anti-automorphism of E, then every left-ideal in E is a right-ideal, since every right-ideal in the completely primary and uni-serial ring E is a left-ideal; and there exists clearly an anti-automorphism of E_n. Thus the sufficiency of the condition in (b) is an immediate consequence of Corollary 6.3. If there exists a duality of the partially ordered set of all the right-ideals in E_n upon the set of the left-ideals in E_n, then we infer from Corollary 6.3 the existence of an anti-automorphism of E_n; and we deduce from Lemma 5.1 the existence of an anti-automorphism of E.

Remark. It has been pointed out that the hypothesis: $2 < n$ is needed for the proof of the necessity of the conditions enumerated in Corollaries 6.3 and 6.4; but it is clear from the proofs that this hypothesis is not needed for proving the sufficiency of these conditions.

UNIVERSITY OF ILLINOIS,
URBANA, ILL.

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THE LINEAR q-DIFFERENCE EQUATION OF THE SECOND ORDER.* [1]

By JEANNE LE CAINE.

1. The general matrix equation

$$Y\left(\frac{ax+b}{cx+d}\right) = R(x)Y(x)$$

in which Y and R are matrices of order n, leads to a variety of interesting functions. If the linear transformation $(ax+b)/(cx+d)$ has one fixed point, there exists a linear transformation on the independent variable by which this matrix equation can be reduced to the difference equation

$$Y(x+1) = R(x)Y(x).$$

If the transformation $(ax+b)/(cx+d)$ has two distinct fixed points, the general matrix equation can be reduced to a q-difference equation

$$Y(qx) = R(x)Y(x).$$

If the limiting forms of the equation are considered, it contains also the differential equation

$$Y'(x) = R(x)Y(x).$$

The simplest case of the general matrix equation is the case $n = 1$. The difference equation in this case leads to the theory of the gamma function, which is related to the sine function by the equation $\Gamma(x)\Gamma(1-x) = \pi/\sin \pi x$. The q-difference equation leads to the theory of what may be called the q-gamma function. This can be considered the logical "half" of the Weierstrass sigma function on account of the relation

$$\Gamma_q{}^0(1-x) = \Gamma_q{}^\infty(1-x)\, c\, e^{-(\eta/2)t^2-\pi i t}\sigma(t).$$

The differential equation leads to the exponential function.

From the point of view of simplicity the next case to consider is the case

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where $n = 2$, and the matrix $R(x)$ takes the simplified form $(1/p(x))(Ax + B)$ which can be immediately reduced, by means of the case $n = 1$, to the form

$$Y\left(\frac{ax + b}{cx + d}\right) = (Ax + B)Y(x).$$

The linear differential system obtained from this by a limiting process is essentially equivalent to the hypergeometric equation of Gauss. The linear difference system is the hypergeometric difference equation. The q-difference equation in the general form leads to the theory of Heine Series.

The importance of the matrix as a starting point is obvious for differential equations from the work of Schlesinger. A detailed treatment of the hypergeometric difference equation from the matrix point of view is given by Batchelder.[2] The analogous case for q-difference equations has not been treated.

In the present paper, the aim is to treat certain questions of importance in the classification of the q-gamma matrices of the second order, which are the solutions of the q-difference equation

(1) $Y(qx) = (Ax + B)Y(x)$, where A, B are constant matrices of order 2, $|Ax + B| \not\equiv 0$, and $|q|$ is assumed greater than one. Some of the known facts about Heine Series[3] are used as are also the theoretical results which were obtained by Grévy and Leau, Carmichael, Adams, Birkhoff, and Trjitzinsky.[4]

In order to make clear the advantages of the matrix method in this discussion, the paper is divided into two parts. The first develops some

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[2] P. M. Batchelder, *An Introduction to Linear Difference Equations*, Cambridge, 1927.

[3] H. E. Heine, "Untersuchungen über die Reihe

$$1 + \frac{(1-q^\alpha)(1-q^\beta)}{(1-q)(1-q^\gamma)}x + \frac{(1-q^\alpha)(1-q^{\alpha+1})(1-q^\beta)(1-q^{\beta+1})}{(1-q)(1-q^2)(1-q^\gamma)(1-q^{\gamma+1})}x^2 + \cdots,"$$

Journal für Mathematik, vol. 34 (1847), pp. 285-328. E. R. Smith, "Zür Theorie der Heineschen Reihe und ihrer Verallgemeinerung," *Inaugural Dissertation*, München, (1911).

[4] A comprehensive bibliography on the field of difference equations is given by N. E. Nörlund, *Differenzenrechnung*, Berlin, 1924. This has been brought up to the year 1931 by C. R. Adams in an article entitled "Linear q-difference equations," *Bulletin of the American Mathematical Society*, vol. 37 (1931), pp. 361-400. A few references of importance in the following are: C. R. Adams, "On the linear ordinary q-difference equation," *Annals of Mathematics*, (2), vol. 30 (1929), pp. 195-205; G. D. Birkhoff, "The generalized Riemann problem for linear differential equations and the allied problems for linear difference and q-difference equations," *Proceedings of the American Academy of Arts and Sciences*, vol. 49 (1913), pp. 521-568; W. J. Trjitzinsky, "Analytic theory of linear q-difference equations," *Acta Mathematica*, vol. 61 (1933), pp. 1-38.

elementary algebraic theorems regarding the effect of certain types of transformations on equation (1). The second deals with the explicit form of the q-gamma matrices and the relations between them which can be obtained by application of the algebraic theorems of part A.

Among the new results obtained are the following. In **2** a simple normal form for equation (1) is set up and the conditions are obtained under which (1) can be reduced to this form by the transformation $Y(x) = C\bar{Y}(x)$, C a non-singular constant matrix. The following section contains a discussion of necessary and sufficient conditions for the existence of a rational transformation, $Y(x) = L(x)\bar{Y}(x)$, by means of which equation (1) is transformed into a linear equation of the same form.

The explicit forms of the q-gamma matrices which are the fundamental solutions of equation (1) are listed in **6**. In **7** it is shown that if two equations are related by a rational transformation, a rational relation exists between the corresponding fundamental solutions of the equations, and that this is the only case in which such a relation can exist. The position of the family of rationally related equations in the solution of the Riemann Problem for equations in which $|A| \neq 0$ and $|B| \neq 0$ is stated in **7**. In the concluding section it is shown that, by making use of the rational relations between equations, a fundamental solution of an equation for which $|A|$ and $|B|$ are not zero can be represented as a convergent matrix product in which the polynomial part of each factor can be taken to be linear.

A: Transformations on the Matrix Equation.

2. Linear transformations. In the equation

$$Y(qx) = (Ax + B)Y(x) \tag{1}$$

let a_1, a_2 represent the characteristic roots of the matrix A; b_1, b_2 the characteristic roots of the matrix B; α_1, α_2, the zeros of the determinant

$$|Ax + B| = a_1a_2x^2 + (a_{11}b_{22} + b_{11}a_{22} - a_{12}b_{21} - a_{21}b_{12})x + b_1b_2.$$

These six constants will be called the *characteristic constants* of the matrix $Ax + B$, or of equation (1). A matrix (or equation) will be said to be of *type* I if all its characteristic constants are finite and different from zero. The discussion in this and in the following section deals with matrices of type I. The modifications necessary for matrices not of type I will be discussed later.

The characteristic constants of a matrix satisfy the relation $b_1b_2 = a_1a_2\alpha_1\alpha_2$, and so not more than five can be assigned arbitrarily. If a_{12} and b_{12} are simultaneously zero, the matrix (or equation) is said to be reducible, and the constants satisfy the additional conditions $b_1 = -a_1\alpha_1$, $b_2 = -a_2\alpha_2$.

If to equation (1) is applied the linear transformation

(2) $Y(x) = C\bar{Y}(x)$, C a matrix of constants, $|C| \neq 0$, the equation becomes

$$\bar{Y}(qx) = C^{-1}(Ax + B)C\bar{Y}(x) = (C^{-1}ACx + C^{-1}BC)\bar{Y}(x)$$

which is a linear matrix equation of the same form as (1), and with the same characteristic constants. Thus *if two equations are related by a linear transformation* (2) *they have the same characteristic constants.*

Under certain conditions the converse of this is true. This can be stated as follows:

THEOREM I. *If a_1, a_2, b_1, b_2, α_1, α_2, are the characteristic constants of equation* (1) *and A and B do not commute, a non-singular matrix C exists such that the transformation* (2) *applied to equation* (1) *gives*

$$(3)\quad \bar{Y}(qx) = \begin{pmatrix} 0 & a_1(x-\alpha_1) \\ -a_2(x-\alpha_2) & (a_1+a_2)x + b_1 + b_2 \end{pmatrix}\bar{Y}(x) = (\bar{A}x + \bar{B})\bar{Y}(x)$$

or

$$(4)\quad \bar{Y}(qx) = \begin{pmatrix} a_1x + b_1 & 0 \\ 1 & k(a_1x + b_1) \end{pmatrix}\bar{Y}(x).$$

In proving this theorem, there is no loss of generality if $Ax + B$ is assumed to have the form $\begin{pmatrix} a_1x + b_{11} & b_{12} \\ a_{21}x + b_{21} & a_2x + b_{22} \end{pmatrix}$ with $a_{21} = 0$ if $a_1 \neq a_2$, and $a_{21} = 0$ or 1 if $a_1 = a_2$, since there always exists a non-singular matrix C such that $C^{-1}AC = \begin{pmatrix} a_1 & 0 \\ a_{21} & a_2 \end{pmatrix}$. The possibility $a_{21} = 0$, $a_1 = a_2$, is excluded because A cannot be a multiple of the unit matrix since A and B do not commute.

If $b_{12} \neq 0$, the matrix

$$C = \begin{pmatrix} 1 & , & -\dfrac{a_1}{a_2} \\ -\dfrac{b_{11} + a_1\alpha_2}{b_{12}}, & \dfrac{b_{11} + a_1\alpha_2}{b_{12}} & -\dfrac{a_{21}}{a_2} \end{pmatrix},\quad |C| = \{(b_{11} + a_1\alpha_2)(a_2 - a_1) - a_{21}b_{12}\}/b_{12}$$

is non-singular, and $C^{-1}(Ax + B)C = \bar{A}x + B$.

If $b_{12} = 0$, then $b_{11} = b_1 = -a_1\alpha_1$ and $b_{22} = b_2 = -a_2\alpha_2$. The diagonal elements of the matrix $Ax + B$ can be interchanged by a non-singular transformation (2), and if $\alpha_1 \neq \alpha_2$ a further transformation with

$$C = \begin{pmatrix} -1 & , & 1 \\ \dfrac{a_1b_{21} - a_{21}b_2}{a_1^2(\alpha_2 - \alpha_1)}, & -\dfrac{a_2b_{21} - a_{21}b_2}{a_2^2(\alpha_2 - \alpha_1)} & + \dfrac{a_{21}}{a_2} \end{pmatrix}$$

puts $Ax + B$ in the form $\bar{A}x + \bar{B}$. If $\alpha_1 = \alpha_2$, then $a_2 = ka_1$ and $b_2 = kb_1$ and the only case in which A and B do not commute reduces to (4).

That the restrictions made in the above theorem are necessary can be seen by considering the commutativity of $\bar{A}$ and $\bar{B}$. These matrices commute if and only if $b_1 = -a_1\alpha_1$, $b_2 = -a_2\alpha_2$, $\alpha_1 = \alpha_2$. Therefore $C^{-1}AC = \bar{A}$ and $C^{-1}BC = \bar{B}$ are impossible when A and B commute and the conditions $b_1 = -a_1\alpha_1$, $b_2 = -a_2\alpha_2$, $\alpha_1 = \alpha_2$ are not satisfied, and also when these conditions are satisfied and A and B do not commute, as in equation (4).

If the equation takes the form (4), the simple rational transformation

$$Y(x) = \begin{pmatrix} a_1x/q + b_1 & 0 \\ 1 & 1 \end{pmatrix} \bar{Y}(x)$$

yields the equation

$$\bar{Y}(qx) = \begin{pmatrix} a_1x/q + b_1 & 0 \\ k(a_1x + b_1) & k(a_1x + b_1) \end{pmatrix} \bar{Y}(x).$$

This is a linear equation, the characteristic constants of which no longer satisfy the condition $\bar{\alpha}_1 = \bar{\alpha}_2$, and can therefore be transformed by a constant transformation into the corresponding form (3).

In this paper the case where A and B commute will be omitted for the sake of brevity as it can be shown that in this case the solution of equation (1) can be expressed completely in terms of q-gamma functions of the first order, and that the discussion of the equation is analogous to that of first order equations.

In the following, only equations which can be put in the form (3) by a linear transformation will be considered. For such equations the characteristic constants determine the family of equations related to each other by linear transformations. The matrix $\bar{A}x + \bar{B}$ will be called the *normal form* of the matrix $Ax + B$, and equation (3) will be called the *normal member* of the family of equations to which it belongs.

3. Rational transformations on equations of Type I. If the transformation

(5) $\quad Y(x) = L(x)\bar{Y}(x)$, $L(x)$ a rational matrix, $|L(x)| \not\equiv 0$,

is applied to equation (1) the equation becomes

$$\bar{Y}(qx) = L^{-1}(qx)(Ax + B)L(x)\bar{Y}(x).$$

In general, the matrix of this equation is not linear, and the new equation does not fall into the class under consideration. For certain choices of $L(x)$, however, the matrix $L^{-1}(qx)(Ax + B)L(x)$ is again a linear matrix of the form $Cx + D$.

THEOREM II. *If $Ax+B$ is the normal matrix with characteristic constants $a_1, a_2, b_1, b_2, \alpha_1, \alpha_2$ and if $c_1, c_2, d_1, d_2, \beta_1, \beta_2$ are such that*

(6) $c_1 = q^{m_1}a_1$, $c_2 = q^{n_2}a_2$, $d_1 = q^{m_1}b_1$, $d_2 = q^{m_2}b_2$, $\alpha_2 = q^{k_2}\beta_2$, $d_1d_2 = c_1c_2\beta_1\beta_2$ *where m_1, m_2, n_1, n_2, k_2 are arbitrary integers, there exists some non-singular rational matrix $L(x)$ such that $L^{-1}(qx)(Ax+B)L(x)$ is the normal matrix with characteristic constants $c_1, c_2, d_1, d_2, \beta_1, \beta_2$.*

First consider the cases in which one of the integers m_1, m_2, n_1, n_2, k_2 is ± 1, and the others are all zero. These cases fall into three groups.

(i) $m_1 = m_2 = n_1 = n_2 = 0$, $k_2 = \pm 1$.

If $L(x) = \begin{pmatrix} (x/q - \alpha_1)/(x - \alpha_2) & 0 \\ 0 & 1 \end{pmatrix}$, $L^{-1}(qx)(Ax+B)L(x)$ is the normal matrix with characteristic constants $a_1, a_2, b_1, b_2, q\alpha_2, \alpha_1/q$, that is $k_2 = 1$. A simple modification of this matrix gives the rational matrix required for the case $k_2 = -1$.

(ii) $m_1 = m_2 = k_2 = 0$; $n_1 = \pm 1$, $n_2 = 0$ or $n_1 = 0$, $n_2 = \pm 1$.

If $L(x) = \begin{pmatrix} -(a_1x + a_2\alpha_1 + b_1 + b_2)/(x - \alpha_1), & a_1 \\ -a_2(x - \alpha_2)/(x - \alpha_1), & a_2 \end{pmatrix}$, and is non-singular, $L^{-1}(qx)(Ax+B)L(x)$ is the normal matrix with characteristic constants $qa_1, a_2, b_1, b_2, \alpha_1/q, \alpha_2$. This is the case $n_1 = 1$, $n_2 = 0$, and an analogous rational matrix can be found for each of the other cases in this group.

(iii) $n_1 = n_2 = k_2 = 0$; $m_1 = \pm 1$, $m_2 = 0$ or $m_1 = 0$, $m_2 = \pm 1$. To obtain characteristic constants $a_1, a_2, qb_1, b_2, \alpha_2, q\alpha_1$, the case $m_1 = 1$,

$$L(x) = \begin{pmatrix} -(x - q\alpha_1)/\alpha_2 x, & -a_1(x - q\alpha_1)/(b_2 x) \\ a_2(x - q\alpha_1)/b_2 x, & -[qb_1\alpha_2 + b_2 x + (a_1 + a_2)\alpha_2 x]/b_2\alpha_2 x \end{pmatrix}$$

and similar matrices can be found for the other cases.

The matrix $L(x)$ in (i) is always non-singular. The matrices $L(x)$ in (ii) and (iii) will be non-singular unless $Ax+B$ is reducible, that is $b_1 = -a_1\alpha_1$ and $b_2 = -a_2\alpha_2$. In this case a satisfactory non-singular matrix $L(x)$ can be found as the product $L_1(x)L_2(x)L_3(x)$ where $L_1(x)$ is chosen so that $L_1^{-1}(qx)(Ax+B)L_1(x) = A_1x + B_1$ has characteristic constants $a_1, a_2, b_1, b_2, \alpha_1/q, \alpha_2 q$ and is irreducible; $L_2(x)$ is the appropriate matrix of group (ii) or (iii) for A_1x+B_1 so that for the matrix $L_2^{-1}(qx)(A_1x+B_1)L_2(x) = A_2x + B_2$ the integers m_1, m_2, n_1, n_2 have the required values, but $k_2 = 1$; $L_3(x)$ is a matrix of group (i) such that for $L_3^{-1}(qx)(A_2x+B_2)L_3(x)$ the value of k_2 is zero.

It is now clear that the normal matrix with characteristic constants $c_1, c_2, d_1, d_2, \beta_1, \beta_2$ satisfying conditions (6) can be obtained from the normal matrix $Ax + B$ by a finite sequence of transformations of the forms discussed under (i), (ii) and (iii) above. Thus the $L(x)$ of the theorem can be obtained as the product of a finite number of non-singular rational matrices.

It can also be shown that if a linear equation $\bar{Y}(qx) = (Cx + D)\bar{Y}(x)$ can be derived from $Y(qx) = (Ax + B)Y(x)$ by a transformation of the form (5), then the characteristic constants of these equations must satisfy conditions (6).

THEOREM III. *If for some $L(x)$, $L^{-1}(qx)(Ax + B)L(x) = Cx + D$, then*

(6) $c_1 = q^{n_1}a_1$, $c_2 = q^{n_2}a_2$, $d_1 = q^{m_1}b_1$, $d_2 = q^{m_2}b_2$, $\beta_2 = q^{k_2}\alpha_2$, $d_1d_2 = c_1c_2\beta_1\beta_2$ *where m_1, m_2, n_1, n_2, k_2 are integers, $a_1, a_2, b_1, b_2, \alpha_1, \alpha_2$ are the characteristic constants of $Ax + B$ and $c_1, c_2, d_1, d_2, \beta_1, \beta_2$ are the characteristic constants of $Cx + D$.*

In order to establish this it is more convenient to write

$$L^{-1}(qx)(Ax + B)L(x) = Cx + D$$

in the form

$$(Ax + B)P(x)r(qx) = P(qx)(Cx + D)r(x),$$

or

$$(7)\quad (Ax+B)\{P_nq^mx^{n+m} + (P_nr_{m-1} + qP_{n-1})q^{m-1}x^{n+m+1} + \cdots + (r_0P_1 + r_1P_0q)x + r_0P_0\}$$
$$= \{P_nq^nx^{n+m} + (P_nr_{m-1}q + P_{n-1})q^{n-1}x^{n+m-1} + \cdots + (r_0qP_1 + r_1P_0)x + r_0P_0\}(Cx + D)$$

where the rational matrix $L(x)$ is replaced by $P(x)/r(x)$, $P(x)$ a polynomial matrix $P_0 + P_1x + P_2x^2 + \cdots + P_nx^n$, and $r(x)$ a polynomial $r_0 + r_1x + \cdots + r_{m-1}x^{m-1} + x^m$.

Let the zeros of $P(x)$ be $p_1, \cdots, p_{2n}$ and the zeros of $r(x)$ be $\gamma_1, \cdots, \gamma_m$. The determinant of the left hand side of (7) is a polynomial with zeros $\alpha_1, \alpha_2, p_1, \cdots, p_{2n}, \gamma_1/q, \cdots, \gamma_m/q$ and the determinant of the right hand side is a polynomial with zeros $\beta_1, \beta_2, p_1/q, \cdots, p_{2n}/q, \gamma_1, \cdots, \gamma_m$. There is no loss of generality if all the constants p_i, γ_i are assumed not equal to zero. These two sets must be identical. That is, α_1 must equal $\beta_1, \beta_2, p_i/q$ for some $i = 1, \cdots, 2n$ or γ_i for some $i = 1, \cdots, m$. If $\alpha_1 = \beta_1$, the required relation between α_1 and β_1 is established. If $\alpha_1 = p_1/q$, then since p_1 cannot equal p_1/q, $p_1 = \beta_1, \beta_2$, or p_j/q for some $j = 2, \cdots, 2n$ or γ_j for some $j = 1, \cdots, m$. If $p_1 = \beta_1$, then $\alpha_1 = \beta_1/q$ and the required relation is established. Continuing in this way it is evident that for some k_1 and k_2, $\alpha_1 = q^{k_1}\beta_1$ and $\alpha_2 = q^{k_2}\beta_2$.

The remaining relationships can be established from the set of matrix equations obtained by equating coefficients of like powers of x in (6), that is $Br_0P_0 = r_0P_0D$

$$B(r_0P_1 + P_0qr_1) + AP_0r_0 = r_0P_0C + (r_0qP_1 + r_1P_0)D$$
$$B(r_1P_2 + P_1qr_2) + A(r_0P_1 + r_1P_0q) = (r_0P_1q + r_1P_0)C + (r_1qP_2 + r_2P_1)D.$$
. . . . . .

It can be assumed that P_0 is not the zero matrix. If $|P_0| \neq 0$, the first equation implies that $b_1 = d_1$, and $b_2 = d_2$. If $|P_0| = 0$, by multiplying each side of equation (7) on the right by a suitable constant matrix E and on the left by its inverse, P_0 can be reduced to one of the forms $\begin{pmatrix} p_0 & 0 \\ 0 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, $p_0 \neq 0$. In the first case the first equation gives $\bar{b}_{11}p_0 = p_0\bar{d}_{11}$, $\bar{b}_{21} = 0$, and $\bar{d}_{12} = 0$, whence $b_1 = d_1$. The second equation now gives $r_0b_2p_{22} = r_0qp_{22}d_2$, that is, if $p_{22} \neq 0$, $b_2 = qd_2$. Otherwise the investigation must be carried further, but a continuation of a similar argument gives the desired result.

According to these results, there is associated with each equation (1) a family of equations, which consists of all those equations which can be obtained from (1) by transformations of the form (4). Such a family of rationally related equations is characterized by a set of characteristic constants $q^{n_1}a_1$, $q^{n_2}a_2$, $q^{m_1}b_1$, $q^{m_2}b_2$, $q^{k_1}\alpha_1$, $q^{k_2}\alpha_2$ where $b_1b_2 = a_1a_2\alpha_1\alpha_2$, and m_1, m_2, n_1, n_2, k_1, k_2 are arbitrary integers subject to the relation $m_1 + m_2 = n_1 + n_2 + k_1 + k_2$.

4. Transformations on the independent variable. Another way in which the equation

$$Y(qx) = (Ax + B)Y(x) \tag{1}$$

can be transformed into an equation of the same form is by the transformations on the independent variable, $x = kx'$, $k \neq 0$, and $x = 1/x'$. If $x = kx'$, equation (1) becomes

$$Y(qkx') = (Akx' + B)Y(kx') \quad \text{or} \quad Z(qx') = (Akx' + B)Z(x'),$$

where $Z(x') = Y(kx')$. This is an equation with characteristic constants ka_1, ka_2, b_1, b_2, α_1/k, α_2/k.

If $x = 1/x'$ equation (1) becomes $Y(q/x') = (A/x' + B)Y(1/x')$ or $Y(1/x') = (x'/|Ax' + B|)(B'x + A')Y(q/x')$, where B' is the adjoint of B and A' is the adjoint of A. If $Y(q/x') = \Gamma_q\{x'/|Ax' + B|\}Z(x')$ where $\Gamma_q\{x'/|Ax' + B|\}$ is a solution of the first order equation $y(qx')$

$= \{x'/|\, Ax' + B\,|\}y(x')$, equation (1) becomes $Z(qx') = (B'x' + A')Z(x')$. This is an equation of the same form as (1), with characteristic constants b_1, b_2, a_1, a_2, $1/\alpha_1$, $1/\alpha_2$. It should be noted that this discussion is not restricted to equations of type I.

5. Equations of Type II. An equation (or matrix) is said to be of type II if $a_1a_2b_1b_2 = 0$. Equations of this type can be divided into three groups: Type IIA, with $b_1b_2 = 0$, $a_1a_2 \neq 0$, and hence α_1, α_2 are finite and $\alpha_1\alpha_2 = 0$; Type IIB, $a_1a_2 = 0$, $b_1b_2 \neq 0$, and $\alpha_1\alpha_2 = \infty$; Type IIC, $b_1b_2 = 0$, $a_1a_2 = 0$, so that $\alpha_1 = 0$, and $\alpha_2 = \infty$. For equations of type IIA it can be shown exactly as in **2** that if A and B do not commute a non-singular matrix C exists such that the transformation (2) applied to the equation puts it in the normal form (3). The form (4) is not possible under the condition of type IIA.

For equations of type IIB a modified form of equations (3) must be used as normal form since some of the elements in the matrix are indeterminate. It is unnecessary to consider separately matrices of type IIB since by a transformation on the independent variable of the form $x = 1/x'$ as discussed in the preceding section, such an equation can be transformed into an equation of type IIA.

For equations of type IIC ($a_2 = 0$, $b_2 = 0$) the matrix $Ax + B$ can be put into the form $\begin{pmatrix} a_1x + b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$ if $a_1 \neq 0$, and $|\, Ax + B\,| = a_1b_{22}x$. The normal form in this case is $\begin{pmatrix} 0 & a_1x \\ -b_{22} & a_1x + b_1 \end{pmatrix}$. If $a_1 = 0$, b_1 cannot be zero, and by a transformation of the form $x = 1/x'$ this can be transformed into the preceding case.

Families of rationally related equations can be set up for equations of type II as they were in **3**. The second theorem of **3** applies without modification. The first theorem of **3** can be established by exhibiting matrices $L(x)$ similar to those for type I. Since multiplying the characteristic constants by powers of q cannot affect the type of the equation, rational relations can exist only between equations of the same type.

B: Relations Between Solutions.

6. Fundamental solutions. The matrix equation (3) is equivalent to the second order equation with linear coefficients

(8) $a_1(x - \alpha_1)y(q^2x) - [(a_1 + a_2)x + b_1 + b_2]y(qx) + a_2(x - \alpha_2)y(x) = 0$,

in the sense that each can be derived from the other. A matrix solution of

(3) can be built up from an independent set of solutions of (8), that is a pair of solutions $y_1(x)$, $y_2(x)$ such that $y_1(x)y_2(qx) - y_2(x)y_1(qx) \neq 0$. The definition of type given for equations of the form (3) can also be applied to equation (8). The forms of the solutions of (8) and therefore of (3) are essentially different for equations of type I and type II.

Heine [5] showed that the general series

$$\phi(\alpha,\beta,\gamma,q,x) = 1 + \sum_{i=1}^{\infty} \frac{(1-q^{\alpha})(1-q^{\alpha+1})\cdots(1-q^{\alpha+\nu-1})(1-q^{\beta})\cdots(1-q^{\beta+\nu-1})}{(1-q)(1-q^2)\cdots(1-q^{\nu})(1-q^{\gamma})\cdots(1-q^{\gamma+\nu-1})}$$

satisfies a particular second order equation. Any second order equation (8) of type I with the additional condition that $a_1 \neq q^{n_1}a_2$, $b_1 \neq q^{n_2}b_2$, n_1, n_2 integers, can be reduced to this equation. An equation of type I for which $a_1 \neq q^{n_1}a_2$, $b_1 \neq q^{n_2}b_2$, n_1, n_2 integers, will be said to be of type Ia, otherwise of type Ib. Smith [6] obtained, in terms of Heine series, two fundamental systems of solutions for an equation of type Ia, one valid in the neighborhood of zero, the other valid in the neighborhood of infinity. He also discussed the solutions of an equation (8) of type Ib. His results can be obtained more simply by using the method given by Adams [7] for equations of order n. Using the notation

$$\alpha = \frac{\log a_2 - \log b_2}{\log q} + \frac{\log(-\alpha_2)}{\log q}, \qquad \beta = \frac{\log a_1 - \log b_2}{\log q} + \frac{\log(-\alpha_2)}{\log q}$$

$$\gamma = \frac{\log b_1 - \log b_2}{\log q} + 1, \qquad t = \frac{\log x}{\log q},$$

the fundamental matrix solutions for equations of type Ia are:

$$\Gamma_q^{0}(x; A, B) = E(x) \begin{pmatrix} {b_1}^t & 0 \\ 0 & {b_2}^t \end{pmatrix} \begin{pmatrix} k_1 & 0 \\ 0 & k_2 \end{pmatrix} e^{\pi i t}$$

and

$$\Gamma_q^{\infty}(x; A, B) = G(x) \begin{pmatrix} {a_1}^t & 0 \\ 0 & {a_2}^t \end{pmatrix} \begin{pmatrix} k_1 & 0 \\ 0 & k_2 \end{pmatrix}.$$

Here k_1, k_2 are arbitrary constants. The matrix $E(x)$ is equal to

$$\Gamma_q^{0}(1-x/\alpha_1) \times \begin{pmatrix} \phi(\alpha,\beta,\gamma,q,x/\alpha_2), & \phi(\alpha-\gamma+1,\beta-\gamma+1,2-\gamma,q,x/\alpha_2 \\ -\frac{b_1}{a_1\alpha_1}\phi(\alpha,\beta,\gamma,q,qx/\alpha_2), & -\frac{b_2}{a_1\alpha_1}\phi(\alpha-\gamma+1,\beta-\gamma+1,2-\gamma,q,qx/\alpha_2 \end{pmatrix}$$

[5] H. E. Heine, *loc. cit.*
[6] E. R. Smith, *loc. cit.*
[7] C. R. Adams, *loc. cit.*

and is a matrix of entire functions. The matrix $G(x)$ is equal to

$$\Gamma_q^{\infty}(1-\alpha_1/x) \times \begin{pmatrix} \phi(\alpha, \alpha-\gamma+1, \beta-\gamma+1, q, \alpha_1 q^2/x), & \phi(\beta, \beta-\gamma+1, \beta-\alpha+1, q, \alpha_1 q^2/x) \\ \phi(\alpha, \alpha-\gamma+1, \beta-\gamma+1, q, \alpha_1 q/x), & \phi(\beta, \beta-\gamma+1, \beta-\alpha+1, q, \alpha_1 q/x) \end{pmatrix}.$$

It is a matrix of meromorphic functions and is analytic in the neighborhood of infinity.

For an equation (3) of type Ib $\Gamma_q^0(x; A, B)$ is as above unless $b_1 = q^{n_2} b_2$. in which case $\Gamma_q^0(x; A, B) = b_1{}^t e^{\pi i t} S_0(x) \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$ where $S_0(x)$ is a matrix of power series in x analytic at $x = 0$; $\Gamma_q^{\infty}(x; A, B)$ is as above unless $a_1 = q^{n_1} a_2$, when $\Gamma_q^{\infty}(x; A, B) = a_1{}^t q^{\frac{1}{2}(t^2-t)} S_{\infty}(x) \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$ where $S_{\infty}(x)$ is a matrix of power series in $1/x$, analytic at $x = \infty$.

Formal matrix solutions for equations of type II can be obtained by the same method. For type IIA, $\Gamma_q^{\infty}(x; A, B) = q^{\frac{1}{2}(t^2-t)} G(x) \begin{pmatrix} a_1{}^t, & 0 \\ 0, & a_2{}^t \end{pmatrix}$ where $G(x)$ is a matrix of power series in $1/x$ which converge everywhere except for $x = 0$. The fundamental solutions about zero have the following forms:

If $b_2 = 0, b_1 \neq 0, \alpha_1 \neq 0$: $\Gamma_q^0(x; A, B) = S_1(x) \begin{pmatrix} b_1{}^t & 0 \\ 0 & (a_1 a_2 \alpha_1 / b_1)^t q^{\frac{1}{2}(t^2-t)} \end{pmatrix} e^{\pi i t}$.

If $b_2 = 0, b_1 \neq 0, \alpha_1 = 0$: $\Gamma_q^0(x; A, B) = S_2(x) \begin{pmatrix} b_1{}^t & 0 \\ 0 & (a_1 a_2 / b_1)^t q^{(t^2-t)} \end{pmatrix}$.

If $b_1 = 0, b_2 = 0, \alpha_1 \neq 0$: $\Gamma_q^0(x; A, B) = q^{\frac{1}{4}(t^2-t)} S_3(x) \begin{pmatrix} (a_1 a_2 \alpha_1)^{\frac{1}{2}t} & 0 \\ 0 & [-(a_1 a_2 \alpha_1)^{\frac{1}{2}}]^t \end{pmatrix} e^{\pi i t}$.

type IIC, $\Gamma_q^{\infty}(x; A, B) = S(x) \begin{pmatrix} a_1{}^t q^{\frac{1}{2}(t^2-t)} & 0 \\ 0 & (-b_{22})^t \end{pmatrix}$.

If $b_1 \neq 0$, $\Gamma_q^0(x; A, B) = S_4(x) \begin{pmatrix} b_1{}^t & 0 \\ 0 & (-b_{22} a_1 / b_1)^t q^{\frac{1}{2}(t^2-t)} \end{pmatrix}$.

If $b_1 = 0$, $\Gamma_q^0(x; A, B) = S_5(x) \begin{pmatrix} (b_{22} a_1)^{\frac{1}{2}t} & 0 \\ 0 & [-(b_{22} a_1)^{\frac{1}{2}}]^t \end{pmatrix}$.

The matrices $S_1(x)$, $S_2(x)$, $S_4(x)$ are matrices of power series in x; the elements of one column are convergent series, those of the other, divergent series. The elements of $S_3(x)$ and $S_5(x)$ are convergent power series in $x^{\frac{1}{2}}$.

If $\bar{Y}(x)$ is any matrix solution of (1) the general solution of (1) is $\bar{Y}(x) P(x)$, where $P(x)$ is a matrix of q-periodic functions, that is, $P(qx) = P(x)$.[8]

[8] R. D. Carmichael, "The general theory of linear q-difference equations," *American Journal of Mathematics*, vol. 34 (1912), pp. 147-168.

7. Rational relations between fundamental solutions. The rational transformations (5) by which an equation (3) is transformed into another equation of the same form, set up rational relations between the fundamental solutions of these equations. This is because these transformations have the property stated in the following theorem.

THEOREM IV. *If the transformation* $Y(x) = L(x)\bar{Y}(x)$ *transforms the equation* $Y(qx) = (Ax + B)Y(x)$ *into* $\bar{Y}(qx) = (Cx + D)\bar{Y}(x)$, *then*

$$\Gamma_q^0(x; A, B) = L(x)\Gamma_q^0(x; C, D)\begin{pmatrix} r_1 & 0 \\ 0 & r_2 \end{pmatrix}$$

and

$$\Gamma_q^\infty(x; A, B) = L(x)\Gamma_q^\infty(x; C, D)\begin{pmatrix} s_1 & 0 \\ 0 & s_2 \end{pmatrix}$$

where r_1, r_2, s_1, s_2 *are constants.*

Since the two equations belong to the same family of rationally related equations, they must be of the same type and their characteristic values must have the property that $a_1 = q^{n_1}c_1$, $a_2 = q^{n_2}c_2$, $b_1 = q^{m_1}d_1$, $b_2 = q^{m_2}d_2$, $\alpha_2 = q^{k_2}\beta_2$, where a_1, a_2, b_1, b_2, α_1, α_2 are the characteristic constants of $Ax + B$, c_1, c_2, d_1, d_2, β_1, β_2 are the characteristic constants of $Cx + D$, and m_1, m_2, n_1, n_2, k_2 are integers. The relations imply that $b_1{}^t = x^{m_1}d_1{}^t$, $b_2{}^t = x^{m_2}d_2{}^t$, $a_1{}^t = x^{n_1}c_1{}^t$, $a_2{}^t = x^{n_2}c_2{}^t$. From the relationship which exists between the two equations, it follows that $\Gamma_q^0(x; A, B)$ and $L(x)\Gamma_q^0(x; C, D)$ are solutions of the same equation, and therefore $\Gamma_q^0(x; A, B) = L(x)\Gamma_q^0(x; C, D)P(x)$, where $P(x)$ is a q-periodic matrix. If the equations are of type Ia, this relation is $S(x)\begin{pmatrix} x^{m_1} & 0 \\ 0 & x^{m_2} \end{pmatrix}\begin{pmatrix} d_1{}^t & 0 \\ 0 & d_2{}^t \end{pmatrix} = L(x)\bar{S}(x)\begin{pmatrix} d_1{}^t & 0 \\ 0 & d_2{}^t \end{pmatrix}P(x)$, and therefore $P(x)$ must be a diagonal matrix of constants, $\begin{pmatrix} r_1 & 0 \\ 0 & r_2 \end{pmatrix}$. The theorem can be established by a similar argument for equations of other types, and for the fundamental solutions about infinity.

The rational relations which exist between solutions of equations of the same family are the only rational relationships which can exist. If $\Gamma_q^0(x; A, B) = L(x)\Gamma^0{}_q(x; C, D)$ this value can be substituted for $\Gamma_q^0(x; A, B)$ in the relation $\Gamma_q^0(qx; A, B) = (Ax + B)\Gamma_q^0(x; A, B)$, giving

$$L(qx)\Gamma_q^0(qx; C, D) = (Ax + B)L(x)\Gamma_q^0(x; C, D)$$

or, since $\Gamma_q^0(qx; C, D)\Gamma_q^{0-1}(x; C, D) = Cx + D$,

$$L(qx)(Cx + D) = (Ax + B)L(x).$$

Therefore the two equations belong to the same family.

In this way there is set up a characterization of all situations in which a rational relationship exists between two fundamental solutions of a second order equation, where it is assumed that both are about zero, or both about infinity. By looking at individual elements of the matrices involved, interesting rational relationships are obtained which must be satisfied by the series involved in the matrices $S(x;A,B)$ and $S(x;C,D)$. No transcendental term occurs, since the relation between them takes the form $S(x;A,B) = L(x)S(x;C,D)\begin{pmatrix} x^{s_1} & 0 \\ 0 & x^{s_2} \end{pmatrix}$ where s_1, s_2 are integers.

For matrices of type Ia, the series involved are Heine Series and the relations obtained in this way include the known relations satisfied by Heine Series. These were obtained by Heine by formal manipulation suggested by analogy with the similar relationships which exist between hypergeometric series. The approach based on the q-difference equation gives a logical basis for these relationships, and a characterization of all the relations of this form which can exist.

8. The Riemann problem. The q-periodic matrix which has the property that $\Gamma_q{}^0(x;A,B) = \Gamma_q{}^\infty(x;A,B)P(x)$ is called the fundamental periodic matrix for equation (1). For the equation of the first order, $y(qx) = (1-x)y(x)$, the fundamental periodic function has been explicitly determined.[9] This function is

$$p(x) = \Gamma_q{}^0(1-x)/\Gamma_q{}^\infty(1-x) = ce^{-(\eta/2)t^2-\pi it}\sigma(t)$$

where $\sigma(t)$ is the Weierstrass sigma function belonging to the periods $\omega = 1$, $\omega' = 2\pi i/\log q$, and $c = (1-1/q)^2(1-1/q^2)^2\cdots$.

If (1) is of type Ia the explicit form of its fundamental matrix can be obtained by making use of the asymptotic properties of the Heine Series.[10] The result, expressed in terms of the fundamental periodic function for equations of order one is $P(x;A,B) = (p(-a_ix/b_j)\cdot c_{ij})$, $(i,j=1,2)$, where

$$c_{11} = \prod_{\nu=0}^{\infty} \frac{(1+a_1\alpha_1q^\nu/b_2)(1+b_1q^{\nu+1}/a_2\alpha_2)}{(1-b_1q^{\nu+1}/b_2)(1-a_1q^\nu/a_2)}, \quad c_{21} = \prod_{\nu=0}^{\infty} \frac{(1+a_2\alpha_2q^\nu/b_2)(1+b_1q^{\nu+1}/a_1\alpha_2)}{(1-b_1q^{\nu+1}/b_2)(1-a_2q^\nu/a_1)}$$

$$c_{12} = \prod_{\nu=0}^{\infty} \frac{(1+a_1\alpha_2q^\nu/b_1)(1+b_2q^{\nu+1}/a_2\alpha_2)}{(1-b_2q^{\nu+1}/b_1)(1-a_1q^\nu/a_2)}, \quad c_{22} = \prod_{\nu=0}^{\infty} \frac{(1+a_2\alpha_2/b_1q^\nu)(1+b_2q^{\nu+1}/a_1\alpha_2)}{(1-b_2q^{\nu+1}/b_1)(1-a_2q^\nu/a_1)}.$$

The Riemann problem has been stated and solved for q-difference equations of order n, with properties corresponding to those of type Ia as defined here.[11] For q-difference equations of the second order the results can be

[9] Birkhoff, *loc. cit.*, p. 564.

[10] Smith, *loc. cit.*

[11] Birkhoff, *loc. cit.*, pp. 559-568.

obtained in a more precise form, and without using the limiting process employed in the general case. The method by which this is done is indicated below.

Assume that two matrices $Y_0(x)$ and $Y_\infty(x)$ are defined as follows:

$$Y_0(x) = B(x) \begin{pmatrix} b_1^t & 0 \\ 0 & b_2^t \end{pmatrix} e^{\pi i t}$$

$$Y_\infty(x) = A(x) \begin{pmatrix} a_1^t & 0 \\ 0 & a_2^t \end{pmatrix} q^{\frac{1}{2}(t^2-t)}$$

where $B(x)$ is analytic throughout the finite plane, $|B(0)| \neq 0$, so that $Y_0(x)$ is analytic for $x \neq 0, \infty$; $A(x)$ is analytic at $x = \infty$, $|A(\infty)| \neq 0$, and $A(x)$ is analytic except for poles throughout the finite plane. Let $P(x)$ be defined by the equation $Y_0(x) = Y_\infty(x)P(x)$. Assume that $P(x)$ has the property that $P(qx) = P(x)$, and that there exists some loop L about $x = 0$ which cuts the spirals $\theta = c + \arg q/\log|q|$ only once, and is such that $Y_\infty(x)$ is analytic outside this loop ($x \neq \infty$) and $|Y_\infty(x)| \neq 0$ outside this loop. Then $P(x)$ is analytic outside the loop L, and therefore because of the relation $P(qx) = P(x)$, $P(x)$ is analytic throughout the plane, $x \neq 0, \infty$. From the definition of $P(x)$ it follows that

$$p_{ij}(x) = e^{(\log b_j - \log a_i)t} q^{\frac{1}{2}(t^2-t)} e^{-\pi i t} \phi_{ij}(x)$$

where $\phi_{ij}(x)$ is analytic, and therefore $p_{ij}(x) = c_{ij}p(-a_i x/b_j)$ where $p(x)$ is the fundamental periodic function for equations of order one.

Since $Y_0(x)$ is unaltered in form if it is replaced by $Y_0(x)\begin{pmatrix} c_1 & 0 \\ 0 & c_2 \end{pmatrix}$ and similarly $Y_\infty(x)$ may be replaced by $Y_\infty(x)\begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix}$, the coefficients c_{ij} are not uniquely determined, but can be replaced by $c_j c_{ij}/d_i$. This does not alter the zeros of $|P(x)|$ which are the solutions of the equation

$$\frac{p(-a_1x/b_1)p(-a_2x/b_2)}{p(-a_1x/b_2)p(-a_2x/b_1)} = \frac{c_{12}c_{21}}{c_{11}c_{22}}.$$

The zeros of $|P(x)|$ occur in two sets of the form $q^n\alpha$ ($n = 0, \pm 1, \pm 2, \cdots$). If α_1 is a member of one set of zeros, a member of the second set is given by $\alpha_2 = b_1b_2/a_1a_2\alpha_1$. This follows from the property $p(x) = p(1/x)$. Thus the equation $|P(x)| = 0$ determines transcendentally two sets $q^n\alpha_1$, and $q^n\alpha_2$, ($n = 0, \pm 1, \pm 2, \cdots$). These lie on the two spirals

$$S_1: \ \theta - \arg \alpha_1 = \frac{\arg q}{\log|q|}(\log r - \log|\alpha_1|)$$

and

$$S_2\colon\ \theta - \arg \alpha_2 = \frac{\arg q}{\log |\, q \,|}\,(\log r - \log |\, \alpha_2 \,|).$$

If $\log b_1$, $\log b_2$, $\log a_1$, $\log a_2$, and $P(x)$ are assigned in advance, where $P(x)$ is subject to the restrictions $p_{ij}(x) = c_{ij}p(-a_ix/b_j)$, there exists a matrix equation with $Y_0(x)$, $Y_\infty(x)$ as matrix solutions. For any loop about $x=0$, not passing through $|\,P(x)\,| = 0$, which has the property that it cuts the spirals S_1 and S_2 only once, there exist matrices $Y_0(x)$ and $Y_\infty(x)$ with the property that $|\,Y_0(x)\,| \neq 0$ within and along the loop, and the elements of $Y_\infty(x)$ are analytic and $|\,Y_\infty(x)\,| \neq 0$ without the loop. Matrices having these properties are given by the fundamental matrix solutions of some equation belonging to the family with characteristic constants $q^{n_1}a_1$, $q^{n_2}a_2$, $q^{m_1}b_1$, $q^{m_2}b_2$, $q^{k_1}\alpha_1$, $q^{k_2}\alpha_2$, $m_1 + m_2 = n_1 + n_2 + k_1 + k_2$. If the loop has the additional property that if it cuts the spiral S_1 between the points α_1 and $q\alpha_1$, it cuts S_2 between α_2 and $q\alpha_2$, the matrices having these properties are the fundamental solutions of the equation with characteristic constants a_1, a_2, b_1, b_2, α_1, α_2.

9. Expression of the fundamental solutions as matrix products. The q-gamma matrices $\Gamma_q{}^0(x; A, B)$ and $\Gamma_q{}^\infty(x; A, B)$ which are solutions of equations of type I can be expressed as convergent infinite products of matrices. This infinite product representation is analogous to that for gamma functions of the first order. The method is indicated below for $\Gamma_q{}^0(x; A, B)$, which will be denoted by $\Gamma_0(x)$ for simplicity.

The equation $Y(qx) = (Ax + B)Y(x)$ is equivalent to

$$Y(x) = (Ax/q + B)(Ax/q^2 + B)\cdots(Ax/q^n + B)Y(x/q^n).$$

In particular,

$$\Gamma_0(x) = (Ax/q + B)(Ax/q^2 + B)\cdots(Ax/q^n + B)\Gamma_0(x/q^n).$$

The matrix $\Gamma_0(x) = S_0(x)T_0(x)$, where $S_0(x)$ is a power series and $T_0(x)$ is a matrix the elements of which are transcendental. Replace $\Gamma_0(x/q^n)$ by $S_{0k}(x/q^n)T_0(x/q^n)$ where $S_{0k}(x/q^n)$ is the matrix obtained by including only the first $k+1$ terms of $S_0(x)$. Consider the limit

$$\lim_{n\to\infty}\Big\{\prod_{m=1}^{n}(Ax/q^m + B)S_{0k}(x/q^n)T_0(x/q^n)\Big\}. \tag{9}$$

If this limit exists, it is equal to $\Gamma_0(x)$ since $\lim_{n\to\infty} S_0(x/q^n)S_0{}^{-1}(x/q^n) = I$. The product can be written in the form

$$S_{0k}{}^{-1}(x)T_0{}^{-1}(x)\prod_{m=1}^{n}[T_0{}^{-1}(x/q^{m-1})S_{0k}{}^{-1}(x/q^{m-1})(Ax/q^m + B)S_{0k}(x/q^m)T_0(x/q^m)].$$

Each factor here is $T_0^{-1}(qz)S_{0k}^{-1}(qz)(Az+B)S_{0k}(z)T_0(z)$, $z=x/q^m$, which can be written as $I+z^{k+1}T_0^{-1}(z)\Psi(z)T_0(z)$, $\Psi(z)$ a matrix of power series, since $S_{0k}(z)T(z)$ satisfies the equation up to terms of the k-th degree. Therefore the matrix product for $\Gamma_0(x)$ is

$$\Gamma_0(x)=S_{0k}^{-1}(x)T_0^{-1}(x)\prod_{m=1}^{\infty}[I+(x/q^m)^{k+1}T_0^{-1}(x/q^m)\Psi(x/q^m)T_0(x/q^m)],$$

which will converge uniformly in a region R if k can be chosen so that the series $\sum_{m=1}^{\infty}(x/q^m)^{k+1}T_0^{-1}(x/q^m)\Psi(x/q^m)T_0(x/q^m)$ converges absolutely in R. For x in some region R, and $m>N$ the terms of the matrix $\Psi(x/q^m)$ are less in absolute value than some constant M; therefore the convergence of the series depends on the form of the matrices $T(x)$ and therefore on the type of the equation. If the equation is of type Ia, $T_0(x)=\begin{pmatrix} b_1{}^t & 0 \\ 0 & b_2{}^t \end{pmatrix}$. The general term in the matrix series is then

$$(x/q^m)^{k+1}\begin{pmatrix} \Psi_{11}(x/q^m), & (b_1/b_2)^{\log(x/q^m)/\log q}\Psi_{12}(x/q^m) \\ (b_2/b_1)^{\log(x/q^m)/\log q}\Psi_{21}(x/q^m), & \Psi_{22}(x/q^m) \end{pmatrix}.$$

But $(b_i/b_j)^{\log(x/q^m)/\log q}=(x/q^m)^{(\log b_i-\log b_j)/\log q}$. Thus the series converges if $\sum_{m=1}^{\infty}(x/q^m)^{k+1+(\log b_i-\log b_j)/\log q}$ converges. If b_1, b_2 lie in the circular ring $|q|^{-\frac{1}{2}}<x\leqq|q|^{\frac{1}{2}}$ then $|b_i/b_j|<|q|$ and

$$|q^{-m(k+1+\log(b_i/b_j)/\log q)}|=|q|^{-n(k+1+\log(b_i/b_j)/\log q)}\leqq|q|^{-nk}.$$

Thus in this case the series converges if $k>0$, that is if $k=1$. For any equation of type Ia, there is some $L(x)$ such that a transformation (5) transforms the equation into an equation of the family for which b_1, b_2 lie in the ring $|q|^{-\frac{1}{2}}<x\leqq|q|^{\frac{1}{2}}$. Thus $\Gamma_0(x)$ for any equation of type Ia can be expressed as $L(x)\Pi(x)$ where $\Pi(x)$ is a product of the form (9) with k chosen equal to 1.

For equations of type Ib, k can always be chosen equal to 1.

SMITH COLLEGE,
NORTHAMPTON, MASSACHUSETTS.

A CLASS OF TRANSFORMATION GROUPS IN E_n.*

By DEANE MONTGOMERY [1] and LEO ZIPPIN.

1. Compact transformation groups in E_n (euclidean n-space) which have at least one $n-1$ dimensional orbit appear to be of considerable interest, and we study such groups here. It is known [6] that with a single exception all orbits of such a group are $n-1$ dimensional. We show that this one exceptional orbit consists of a single point, that is, that it is a stationary point of the group. Our principal conclusion is that there is a "ray" which forms a cross section for the orbits and that G acts on E_n in a way which is determined by its action on one $n-1$ dimensional orbit and by this ray.

2. We follow the terminology of [4] and [6]. We assume that G is a compact connected topological group satisfying the second countability axiom, and that to every element g of G there is given a homeomorphism $g(x)$ of E_n into itself in such a way that

(1) $g_1[g_2(x)] = (g_1 g_2)(x)$
(2) $g(x)$ is simultaneously continuous in g and x
(3) if e is the identity of G, $e(x) = x$ for all x.

We assume furthermore that G is effective which means that two distinct elements of G have associated with them two distinct homeomorphisms.

We shall have occasion to use the decomposition space E^*_n whose points are the orbits of G in E_n. The natural transformation L from E_n to E^*_n is continuous and open. We take it as known [6] that if one orbit is $n-1$ dimensional then every orbit is $n-1$ dimensional except for one orbit of dimension less than $n-1$, which we denote by $G(z)$ where z is some point of this orbit. The space E^*_n is homeomorphic to a closed half line and its end point is $L(z)$. It is not immediately obvious that $G(z)$ is a point but this will be verified in what follows. We also take it as known that G must be a Lie group. This is because G acts effectively and with locally connected orbits on the connected locally Euclidean space $E_n - G(z)$ [4, 6].

3. We begin by proving a lemma on finding cross sections. For this lemma it is not necessary to assume that our decomposition is given by a

* Received June 5, 1942.
[1] Guggenheim fellow.

group. In this section, then, we take R to be a compact arc-wise connected metric space which is filled with a continuous collection of compact arc-wise connected sets. In analogy with the situation for groups we denote by $G(x)$ that set of the collection which contains x. The sets of the collection are to be mutually exclusive and by definition the continuity of the collection means that the natural mapping L from R to the decomposition space is continuous and open. We assume that $L(R)$, the decomposition space of R, is homeomorphic to an interval a^*b^* and that the collection enjoys the following property:

(A) For every $\epsilon > 0$ there is a $\delta > 0$ such that if y is in $G(x)$ and $d(x, y)$ is less than δ then x and y may be joined by an arc in $G(x)$ of diameter ϵ.

We have assumed that any two points x and y of R can be joined by an arc and we also assume that this arc can be so chosen that its diameter approaches zero with $d(x, y)$. This assumption is used for the first time in the corollary to Lemma 1.

LEMMA 1. *Assuming the conditions described in the preceding paragraph let p be in $L^{-1}(a^*)$ and q be in $L^{-1}(b^*)$. Then there is a monotonic arc from p to q, that is the homeomorphic image $x(t)$ of an interval such that if $t_1 < t_2$, then $Lx(t_1) \leqq Lx(t_2)$.*

This lemma does not assert that a cross sectioning arc exists but only that there exists an arc which never moves backward. It might of course remain in the same set $G(x)$ throughout an interval of values of t.

In order to prove the lemma we begin by choosing an arc J from p to q. Let $a^*_1, a^*_2, a^*_3, \cdots$ be the interior rational points of the interval a^*b^*. Let p_1 and q_1 be, respectively, the first point and last point in which J meets $L^{-1}(a^*_1)$. Now if p_1 and q_1 are distinct we replace the subarc p_1q_1 of J by an arc (p_1q_1) which lies in the set $L^{-1}(a^*_1)$ and which has diameter less than two times (any fixed multiple greater than one would suffice) the greatest lower bound of the diameters of such arcs. The arc J_1 obtained in this way joins p and q and cuts $L^{-1}(a^*_1)$ in a connected set. If p_1 and q_1 are not distinct we let J_1 be J.

Next let p_2 and q_2 be respectively the first and last points in which J_1 meets $L^{-1}(a^*_2)$. If p_2 and q_2 are distinct we replace the arc p_2q_2 by an arc (p_2q_2) which lies in $L^{-1}(a^*_2)$ and whose diameter is less than two times the greatest lower bound of the diameters of such arcs. We obtain in this way an arc J_2 which joins p and q and meets $L^{-1}(a^*_1)$ and $L^{-1}(a^*_2)$ in connected sets. Proceeding in this way we obtain a sequence of arcs $J_1, J_2, \cdots, J_n, \cdots$ such that J_n meets each of the sets $L^{-1}(a^*_n), L^{-1}(b^*), L^{-1}(a^*_1), \cdots, L^{-1}(a^*_n)$ in a connected set.

The arcs p_nq_n are disjoint subarcs of J and hence the diameter of p_nq_n approaches 0 as n tends to infinity, because an arc can contain at most a finite number of disjoined subarcs whose diameters are greater than a fixed positive number. Because of property (A) and the choice of (p_nq_n) it follows that diameter (p_nq_n) approaches zero.

By definition there is a homeomorphism $h(t)$ taking the interval $(0,1)$ into J. Under this homeomorphism there is a certain interior subinterval I_1 of $(0,1)$ going into the interval p_1q_1. By altering $h(t)$ only on I_1, or not at all if p_1, q_1 are the same point, we obtain a homeomorphism $h_1(t)$ taking (0.1) to J_1. Continuing this process of alterations we obtain a sequence of homeomorphisms $h_1(t)$, $h_2(t), \cdots$ where $h_n(t)$ takes $(0,1)$ to J_n. Because of the remarks in the preceding paragraph the sequence of continuous functions $h_n(t)$ must converge uniformly. Hence they converge to a continuous function $x(t)$. We see also that $x(t)$ is one to one and that it defines a monotonic arc as required in the conclusion of the lemma. Our proof is therefore complete.

COROLLARY. *For each $\epsilon > 0$ there exists a $\delta > 0$ such that if p, q are in R, not necessarily in $L^{-1}(a^*)$ and $L^{-1}(b^*)$, and if $d(p,q) < \delta$, $L(p) \leqq L(q)$, then there exists a monotonic arc from p to q of diameter $< \epsilon$.*

This can be proved by essentially the same argument as the lemma. The first arc J joining p and q can be chosen to have small diameter. By property A the arcs (p_nq_n) can also be chosen with small diameter. Hence the limit arc will have small diameter as we wished to prove.

We shall use the symbol $C(s)$ to denote a finite set of points $a_1, a_2, \cdots, a_k$ in R such that for $i = 1, \cdots, k-1$, we have $L(a_i) < L(a_{i+1})$ and $d(a_i, a_{i+1}) < s$. We speak of this as a chain of mesh less than s joining a_1 and a_k. Note that this finite set is required to be properly monotonic.

LEMMA 2. *If J is a monotonic arc from p to q, $L(p) < L(q)$, and if O is any open set including J then for every $s > 0$ there exists a $C(s)$ joining p and q which is in O.*

We first choose on J a finite set of points $p = b_1, b_2, \cdots, b_k = q$ such that $d(b_i, b_{i+1}) < s/3$, and $L(b_i) \leqq L(b_{i+1})$. With the aid of these points we shall define a chain $C(s) = a_1, \cdots, a_k$. Let $p = a_1$. If $L(b_1) < L(b_2)$ we let $a_2 = b_2$. If not let b_m be the first point b such that $L(a_1) < L(b_m)$. Then $b_1, b_2, \cdots, b_{m-1}$ are all in $G(b_1)$. Now choose $x_2, x_3, \cdots, x_{m-1}$ so that $L(x_2) < L(x_3) \cdots < L(x_{m-1}) < L(b_m)$ and so that the Hausdorff distance from $G(x_2), \cdots, G(x_{m-1})$ to $G(a_1)$ is less than δ where δ is chosen so that the δ neighborhood of J is in O, and so that $\delta < s/3$. We now choose a_i so that a_i is in $G(x_i)$ and so that

group. In this section, then, we take R to be a compact arc-wise connected metric space which is filled with a continuous collection of compact arc-wise connected sets. In analogy with the situation for groups we denote by $G(x)$ that set of the collection which contains x. The sets of the collection are to be mutually exclusive and by definition the continuity of the collection means that the natural mapping L from R to the decomposition space is continuous and open. We assume that $L(R)$, the decomposition space of R, is homeomorphic to an interval a^*b^* and that the collection enjoys the following property:

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We first choose on J a finite set of points $p = b_1, b_2, \cdots, b_k = q$ such that $d(b_i, b_{i+1}) < s/3$, and $L(b_i) \leqq L(b_{i+1})$. With the aid of these points we shall define a chain $C(s) = a_1, \cdots, a_k$. Let $p = a_1$. If $L(b_1) < L(b_2)$ we let $a_2 = b_2$. If not let b_m be the first point b such that $L(a_1) < L(b_m)$. Then $b_1, b_2, \cdots, b_{m-1}$ are all in $G(b_1)$. Now choose $x_2, x_3, \cdots, x_{m-1}$ so that $L(x_2) < L(x_3) \cdots < L(x_{m-1}) < L(b_m)$ and so that the Hausdorff distance from $G(x_2), \cdots, G(x_{m-1})$ to $G(a_1)$ is less than δ where δ is chosen so that the δ neighborhood of J is in O, and so that $\delta < s/3$. We now choose a_i so that a_i is in $G(x_i)$ and so that

$$d(b_i, a_i) < \delta \qquad (i = 2, \cdots, m-1).$$

Choose $a_m = b_m$. The set $a_1, a_2, \cdots, a_m$ is now properly monotonic and has the proper mesh. We continue this process until we come to b_n where b_n is the last b such that $L(b_n) < L(b_k)$. Here we choose a_i's "slanting" from b_n toward b_k entirely analogous to the process already described.

THEOREM 3.1. *Under the hypothesis of Lemma* 1 *there is an arc from* p *to* q *which is a cross section, that is to say an arc from* p *to* q *which intersects each set* $G(x)$ *in one and only one point.*

From the corollary to Lemma 1 and from Lemma 2 it follows that for every positive number ϵ there is a positive number δ such that if p and q are in R, $L(p) < L(q)$, and $d(p, q) < \delta$ then there is for every s a $C(s)$ joining p and q and having diameter less than ϵ.

Let $\epsilon_1, \epsilon_2, \cdots$ be a sequence of positive numbers such that $\epsilon_1 + \epsilon_2 + \cdots$ converges.

Let $\delta_1, \delta_2, \cdots$ be a sequence of positive numbers such that

3.1. $\delta_i < \epsilon_i/3$.

3.2. If p and q are two points in R such that $L(p) < L(q)$ and $d(p, q) < \delta_i$ then for every s there is a $C(s)$ joining p and q of diameter $< \epsilon_i$.

We choose a $C(\delta_1)$ joining p and q, and let its points be

$$a_{11}, a_{12}, \cdots, a_{1k_1}.$$

Then for each i choose a $C(\delta_2)$ of diameter $< \epsilon_1$ joining a_{1i} to $a_{1\,i+1}$. Adding these points (for all i) to those of $C(\delta_1)$ we obtain a $C(\delta_2)$ joining p and q which has the additional property that the finite set between a_1 and $a_{1\,i+1}$ has diameter $< \epsilon_1$. We shall name the points of the set $C(\delta_2)$ as follows:

$$a_{21}, a_{23}, \cdots, a_{2k_2}.$$

We proceed by induction in a natural manner, the general stage being that we have obtained a $C(\delta_{m+1})$ joining p and q and such that the points between a_{mi} and $a_{m\,i+1}$ (of the previous $C(\delta_m)$) have diameter less than ϵ_m. The points of this $C(\delta_{m+1})$ are denoted by

$$a_{m+1\,1}, a_{m+1\,2}, \cdots, a_{m+1\,k_{m+1}}.$$

Let A_m be the points of the $C(\delta_m)$ joining p and q which we find by this process. Clearly A_m is contained in A_{m+1}. We now let A denote the closure of $A_1 + A_2 + \cdots$, and we wish to show that A is an arc forming the desired cross section. In order to prove this it is only necessary to prove that A contains one and only one point on each set $G(x)$.

It is clear that A contains at least one point on each set $G(x)$, that is in each $L^{-1}(x^*)$ for x^* in a^*b^*. Suppose now that y_1 and y_2 are two points of A in a certain set $G(x)$ in R, and let $d(y_1, y_2) = r$. Choose N so large that

$$\epsilon_N + \epsilon_{N+1} + \cdots < r/6.$$

Let a_{Nm_1} be the last point of A_N such that $L(a_{Nm_1})$ precedes $L(x)$ in a^*b^* and let a_{Nm_2} be the first point of A_N such that $L(a_{Nm_2})$ follows $L(x)$ in a^*b^*. Then m_2 is either m_1+1 or m_1+2. In any case all points of $A_1+A_2+A_3+\cdots$ between a_{Nm_1} and a_{Nm_2} have a distance from a_{Nm_1} which is at most equal to $2(\epsilon_N+\epsilon_{N+1}+\cdots)<2r/6$. Therefore these points cannot have both y_1 and y_2 as limit points. Hence A has at most one point on each set $G(x)$ and consequently A is an arc and an arc of the desired kind.

4. We return now to the situation of the introduction and of section **2**, that is to the case where we have a compact connected topological group G acting in E_n and where we assume that some orbit is $n-1$ dimensional. This will be the hypothesis throughout this entire section and the next. In **2** we have already summarized certain conclusions which we may draw from this hypothesis and we shall now draw further conclusions. If H is any topological group we shall denote by H^1 the identity component of H. The symbol G_x denotes the closed subgroup of G leaving x fixed. From [3] we know that if y is near enough to x then G_y is conjugate to a subgroup of G_x.

THEOREM 4.1. *If x and y are in $E_n - G(z)$, then G_x^1 and G_y^1 are conjugate.*

For all x and y in $E_n - G(z)$ the dimension of G_x equals the dimension of G_y. Hence for all y in some neighborhood of a fixed point x, G_y^1 is conjugate to G_x^1 in view of the remark made just before the theorem. Now let p be a definite point of $E_n - G(z)$. We have just shown that the points y such that G_y^1 is conjugate to G_p^1 form an open set. But it can also be seen that this set is closed in $E_n - G(z)$ so that it must be all of $E_n - G(z)$ which concludes the proof.

We consider the integral valued function $\alpha(x)$ defined for every x in $E_n - G(z)$ to have as its value the number of components of G_x. This function is constant on each orbit so that we have an associated function $\alpha^*(x^*)$ defined in $E^*{}_n - z^*$. Both of these functions are lower semi-continuous and this, together with the fact that they are integral valued, implies that they are constant on certain open sets. If $\alpha(x)$ is constant everywhere then G_x and G_y are conjugate for every x and y in $E_n - G(z)$. This may be proved by a method similar to that used in the proof of Theorem 4.1.

THEOREM 4.2. *If x and y are in $E_n - G(z)$ then G_x and G_y are conjugate.*

In view of the remark above it is only necessary to prove that $\alpha(x)$ is constant on $E_n - G(z)$. We have already noted that it is constant on certain open subsets of $E_n - G(z)$.

Consider a point of $E^*_n - z^*$ around which $\alpha^*(x^*)$ is constant. If $\alpha^*(x^*)$ is not constant everywhere then as we move along the ray E^*_n in either one direction or the other we must come to a first point b^* where the function has a different value. We assume, to be definite, that b^* is such that for all points x^* of some interval immediately to the left of b^* we have

$$\alpha^*(x^*) = r$$

whereas

$$\alpha^*(b^*) = s.$$

Then $s = tr$ for some positive integer t greater than one so that we may also write

$$\alpha^*(b^*) = tr.$$

This means roughly that the $n-1$ dimensional manifold $L^{-1}(x^*)$ is wound t times around the $n-1$ dimensional manifold $L^{-1}(b^*)$. Intuition tells us that such a phenomenon can not occur in E_n and this intuitive conjecture can be verified in several ways. In this connection see Eilenberg [2] where a more general situation of this sort is discussed. We now proceed to carry out the verification in a way reminiscent of Eilenberg's method.

Let b be a point of $L^{-1}(b^*)$ and let V be that part of a neighborhood of b which is inside $G(b)$. We assume V to be so small that for x in V, $\alpha(x) = r$. For any x in V we now define a continuous transformation T_x of $G(x)$ into $G(b)$ as follows: Let g_x be an element of G such that

$$g_x G_x g_x^{-1} \subset G_b$$

and assume (see [3]) that g_x approaches the identity as x approaches b. Then if $y = g(x)$ let

$$T_x(y) = g_x g\, g^{-1}(b).$$

This transformation is a local homeomorphism of order t taking $G(x)$ into $G(b)$. Under it no point moves very far and T_x has degree t for properly chosen orientations of $G(x)$ and $G(b)$.

The point z is inside $G(b)$ and also inside $G(x)$ for x in V. These manifolds are orientable and imbedded topologically in E_n. Hence the order of z with respect to them must have absolute value one. But in view of the results about T_x the order of one of them is t times the order of the other which we see is impossible. Hence $\alpha(x)$ is constant everywhere.

THEOREM 4.3. *If a^*b^* is any closed interval of $E^*_n - z^*$ then the orbits of $L^{-1}(a^*b^*)$ have property* (A).

Assuming the lemma to be false there exists a positive ϵ and for each n a pair of points x_n and y_n in the same orbit such that $d(x_n, y_n) < 1/n$ and

x_n and y_n cannot be joined by an arc in their orbit of diameter less than ϵ. We may take it that these pairs of points converge to a point p of $L^{-1}(a^*b^*)$. We now define a homeomorphism T_n of $G(x_n)$ into $G(p)$ much as the local homeomorphism T_x was defined above. Choose a g_n such that

$$g_n G_{x_n} g_n^{-1} = G_p$$

and furthermore make this choice so that g_n approaches the identity of G. For any point $y = g(x_n)$ let

$$T_n(y) = g_n g g_n^{-1}(p).$$

That $T_n(z)$ is uniquely defined and a homeomorphism is not difficult to verify. It is a homeomorphism such that

$$d[y, T_n(y)]$$

approaches zero uniformly as n tends toward infinity.

Hence $T_n(x_n)$ and $T_n(y_n)$ may be joined by an arc in $G(p)$ whose diameter approaches zero when n approaches infinity. Then T_n^{-1} will take these arcs to arcs of small diameter in $G(x_n)$ which join x_n and y_n. We have, therefore, been led to a contradiction which proves the theorem.

5. If a^*b^* is any closed interval of $E^*_n - z^*$ we have now shown that the orbits filling $L^{-1}(a^*b^*)$ fulfill all the conditions imposed at the beginning of section **3**. Consequently there exists an arc which is a cross section for the orbits of $L^{-1}(a^*b^*)$ and which joins any point of $L^{-1}(a^*)$ with any point of $L^{-1}(b^*)$. If x is any point not in $G(z)$ we may, by a repeated application of the process, construct a ray beginning at x which is a cross section for all orbits not inside $G(x)$. The points on $G(x)$ or outside of $G(x)$ form a set which is homeomorphic to the topological product of $G(x)$ and a ray. Any subset of this set may clearly be deformed in a natural way to a position outside of any preassigned bounded set of E_n. We note that during this deformation the set never comes inside $G(x)$, in particular it never comes into contact with $G(z)$. For this process we may choose x as near to $G(z)$ as we like so long as we choose it outside of $G(z)$.

THEOREM 5.1. *The orbit $G(z)$ is a point.*

The fact that G is a Lie group (section **2**) implies that if $G(z)$ is not a point it is a manifold of some dimension p, and therefore it carries a cycle modulo 2 of dimension p. This cycle must be linked with a cycle in the complement of $G(z)$ by the Alexander duality theorem. Thus we see that there is a cycle C in $E_n - G(z)$ which does not bound in $E_n - G(z)$. As we have remarked this cycle may be deformed to a position outside any

bounded set in E_n, for instance it may be deformed to a position outside some $n-1$ sphere including $G(z)$. In this position the cycle bounds outside of $G(z)$ and hence in its original position it must also have bounded outside of $G(z)$. Hence $G(z)$ is a point, namely the point z.

THEOREM 5.2. *There exists a ray beginning at z which is a cross section for all the orbits in E_n.*

We have already seen that we may construct a ray cross section for the orbit $G(x)$ and all exterior orbits when x is not in $G(z)$. Starting with such a cross section we choose a point x_1 not z but inside $G(x)$. We now add to the first cross section an arc x_1x which cross sections the orbits $G(x_1)$ and $G(x)$ and all orbits in between. Continuing in this way we obtain an increasing sequence of rays whose initial points approach z. This sequence of rays approaches a ray which is a cross section of all the orbits in E_n.

THEOREM 5.3. *If x is not z then any cycle of dimension $< n-1$ in $E_n - G(x)$ bounds in $E_n - G(x)$.*

This is because any cycle in $E_n - G(x)$ may be deformed either to z or toward infinity without touching $G(x)$.

This theorem implies that if x is not z, then $G(x)$ has the same homology properties as an $n-1$ sphere, at least where we use such coefficients in $G(x)$ as for instance the reals mod 1, the integers, or the integers mod m.

Since G_x remains constant up to conjugates we have justified our remark in the introduction that the action of G is determined by its action on one orbit and by the ray cross section.

Our results also enable us to see that for any x the image of a sphere of dimension less than $n-1$ in $G(x)$ can be shrunk to a point in $G(x)$.

INSTITUTE FOR ADVANCED STUDY,
QUEENS COLLEGE.

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ON THE DISTRIBUTION OF VALUES OF TRIGONOMETRIC SUMS WITH LINEARLY INDEPENDENT FREQUENCIES.*

By M. KAC.

1. Let

$$f(t) = \sum_{k=1}^{n} a_k \cos 2\pi\lambda_k t, \tag{1}$$

where the a's are real and the λ's real and linearly independent. Denote by $N_T(a)$ the number of t's in $(-T, T)$ for which $f(t) = a$ and consider the ratio $N_T(a)/2T$. The problem of finding

$$E(a) = \lim_{T\to\infty} \frac{N_T(a)}{2T} \tag{2}$$

was considered by E. R. van Kampen, A. Wintner and the present author [1] in an even more general case than (1). However, with the exception of the case $n = 2$, $a_1 = a_2 = 1$, no explicit formula for $E(a)$ was found and it seemed worthwhile to provide such a formula. In what follows we shall prove that

$$E(a) = \frac{1}{2\pi^2} \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \cos a\xi \, \frac{\prod_{k=1}^{n} J_0(|a_k|\xi) - \prod_{k=1}^{n} J_0(|a_k|\sqrt{\xi^2 + 4\pi^2\lambda_k^2\eta^2})}{\eta^2} \, d\eta d\xi. \tag{3}$$

2. Let μ be an arbitrary positive number and put

$$F = F(t; P) = F(t; \phi_1, \cdots, \phi_n) = -a + \sum_{k=1}^{n} a_k \cos 2\pi(\lambda_k t + \phi_k) \quad (0 \leqq \phi_k < 1)$$

$$F' = \partial F/\partial t.$$

Denote by $G_a(\phi_1, \cdots, \phi_n)$ the number of t's in $(0, \mu)$ for which $F = 0$. Then by section **2** of *loc. cit.*[1] if $G_a(\phi_1, \cdots, \phi_n)$ is Riemann integrable the limit (2) exists and

$$E(a) = \frac{1}{\mu} \int_0^1 \cdots \int_0^1 G_a(\phi_1, \cdots, \phi_n) \, d\phi_1 \cdots d\phi_n.^2 \tag{4}$$

We first prove that G_a is Riemann integrable.

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[1] *American Journal of Mathematics*, vol. 61 (1939), pp. 985-991, in particular, sections **2** and **3**. The $(\lambda t + \phi)$'s in formula (12) should be replaced by $2\pi(\lambda t + \phi)$'s.

[2] The right side of (4) does not depend on μ. In *loc. cit.*[1] we put $\mu = 1/\lambda_1$ (assuming $\lambda_1 > 0$) in which case the integral (4) can be reduced to an $(n-1)$-dimensional one.

I. $G_a(\phi_1, \cdots, \phi_n) = G_a(P)$ $(0 \leq \phi_k < 1)$ *is bounded.* If $G_a(P)$ were not bounded we could find such a sequence $\{P_n\}$ of points in the n-dimensional cube that $G(P_n) \to \infty$ as $n \to \infty$. $\{P_n\}$ obviously contains a convergent subsequence which, without loss of generality, may be assumed to be the sequence itself. Let P_0 be the limit of $\{P_n\}$. The functions $F(z; P_n)$ and $F(z; P_0)$ (as functions of the complex variable z) are analytic and $F(z; P_n)$ converges uniformly to $F(z; P_0)$ in every bounded region of the complex z-plane.

Since $F(z; P_0)$ is not identically zero it is a well-known theorem that the number of complex roots of $F(z; P_n)$ in every fixed region approaches the number of complex roots of $F(z; P_0)$ in the same region. This will contradict our assumption that $G_a(P_n) \to \infty$, if we fix the region in such a way as to contain the interval $(0, \mu)$.

II. *The set of points of discontinuity of* $G_a(P)$ *has* n*-dimensional Lebesgue measure* 0. It is easily seen that if P_1 is such that $F(t; P_1)$ $(0 \leq t \leq \mu)$ has single roots only, $G_a(P)$ is continuous at $P = P_1$. Hence, the set of points of discontinuity of $G_a(P)$ is the set H of those P's for which there exists at least one t $(0 \leq t \leq \mu)$ such that simultaneously

$$F(t; P) = 0 \quad \text{and} \quad F'(t; P) = 0.$$

Let $\epsilon > 0$; notice that

$$|F'(t; P)| = 2\pi \left| \sum_{k=1}^{n} a_k \lambda_k \sin 2\pi(\lambda_k t + \phi_k) \right| \leq 2\pi \sum_{k=1}^{n} |a_k \lambda_k| = M_1$$

$$|F''(t; P)| = 4\pi^2 \left| \sum_{k=1}^{n} a_k \lambda_k^2 \cos 2\pi(\lambda_k t + \phi_k) \right| \leq 4\pi^2 \sum_{k=1}^{n} |a_k| \lambda_k^2 = M_2$$

and put $M = \text{Max}(M_1, M_2)$. Put furthermore $t_j = j\epsilon/M$ $(j = 0, 1, \cdots, s)$, where $s = [1 + (M\mu/\epsilon)]$ and observe that $0 = t_0 < t_1 < t_2 < \cdots < t_s \leq \mu$. Let now E_j be the set of those P's for which

$$|F(t_j; P)| < \epsilon \quad \text{and} \quad |F'(t_j; P)| < \epsilon,$$

and let $E = \sum_{j=0}^{s} E_j$.

I assert that $H \subset E$. In fact, let $P_0 \in H$. Then there exists a τ $(0 \leq \tau \leq \mu)$ such that $F(\tau; P_0) = F'(\tau; P_0) = 0$ and we can find such a j that $t_j < \tau \leq t_{j+1}$ $(t_{s+1} = \mu)$. Hence,

$$|F(t_j; P_0)| = |F(t_j; P_0) - F(\tau; P_0)| \leq M_1 |\tau - t_j| < \epsilon$$

and

$$|F'(t_j; P_0)| = |F'(t_j; P_0) - F'(\tau; P_0)| \leq M_2 |\tau - t_j| < \epsilon.$$

This means that $P_0 \in E_j$ and therefore $P_0 \in E$. Thus $H \subset E$ and

$$\text{meas. } H \leq \sum_{j=1}^{s} \text{meas. } E_j.$$

In order to find meas. E_j let us recall that

$$\frac{1}{\pi^2} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \frac{\sin \epsilon\xi \sin \epsilon\eta}{\xi\eta} \exp\{i(\gamma\xi + \delta\eta)\}\, d\xi d\eta$$

is equal to 1 if both $|\gamma|$ and $|\delta|$ are less than ϵ, 0 if either $|\gamma|$ or $|\delta|$ is greater than ϵ, 1/2 if either $|\gamma| = \epsilon$, $|\delta| < \epsilon$ or $|\gamma| < \epsilon$, $|\delta| = \epsilon$, and 1/4 if $|\gamma| = |\delta| = \epsilon$. This leads to the formula

$$\text{meas. } E_j = \frac{1}{\pi^2} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \frac{\sin \epsilon\xi \sin \epsilon\eta}{\xi\eta}$$

$$\times \Big[\int_0^1 \cdots \int_0^1 \exp\{i(\xi F(t_j; P) + \eta F'(t_j; P))\} d\phi_1 \cdots d\phi_n\Big] d\xi d\eta \,^{3}$$

if one notices that the measure of the set of those P's for which $F(t_j; P) = \epsilon$ and the measure of the set of P's for which $F'(t_j; P) = \epsilon$ are equal to 0.[4]

Now, we note that for every t

$$\int_0^1 \cdots \int_0^1 \exp\{i(\xi F + \eta F')\} d\phi_1 \cdots d\phi_n$$

(5)

$$= \exp(-ia\xi) \prod_{k=1}^{n} \int_0^1 \exp\{i(a_k \xi \cos 2\pi(\lambda_k t + \phi_k) + 2\pi a_k \lambda_k \eta \sin 2\pi(\lambda_k t + \phi_k))\} d\phi_k$$

$$= \exp(-ia\xi) \prod_{k=1}^{n} J_0(|a_k| \sqrt{\xi^2 + 4\pi^2 \lambda_k{}^2 \eta^2}),$$

and hence

$$\text{meas. } E_j = \frac{1}{\pi^2} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \frac{\sin \epsilon\xi \sin \epsilon\eta}{\xi\eta}$$

$$\times \exp(-ia\xi) \prod_{k=1}^{n} J_0(|a_k| \sqrt{\xi^2 + 4\pi^2 \lambda_k{}^2 \eta^2})\, d\xi d\eta.$$

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[3] The interchange of the order of integration can be justified in a way exactly analogous to that followed by Kac and Steinhaus in "Sur les fonctions indépendantes III," *Studia Mathematica*, vol. 6 (1936), pp. 93-94.

[4] This statement can be easily proved. It suffices, however, for our purposes to recall that every measurable function $\phi(P)$ has the property that for "nearly all" ϵ's (i. e. except perhaps a denumerable set) the measure of the set of P's for which $\phi(P) = \epsilon$ is equal to zero. In view of this fact we may restrict ourselves to ϵ's not belonging to the exceptional set and save ourselves the trouble of proving that in our particular case the exceptional set is empty.

J_0 denotes here, as usual, the Bessel function of order 0. Since $J_0(|u|) = O(|u|^{-1/2})$ as $|u| \to \infty$ we see that for $n \geq 3$

$$K(\xi, \eta) = \exp(-ia\xi) \prod_{k=1}^{n} J_0(|a_k| \sqrt{\xi^2 + 4\pi^2\lambda_k^2\eta^2}) = O(|\xi|^{-3/4} |\eta|^{-3/4})$$

so that $K(\xi, \eta) \in L^{3/2}$. Hence, by applying the Hölder-Riesz inequality we have

$$\text{meas. } E_j \leq \frac{1}{\pi^2} \left(\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \left| \frac{\sin \epsilon\xi \sin \epsilon\eta}{\xi\eta} \right|^3 d\xi d\eta \right)^{1/3}$$

$$\times \left(\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} |K(\xi, \eta)|^{3/2} d\xi d\eta \right)^{2/3} = \frac{A\epsilon^{4/3}}{\pi^2} \left(\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \left| \frac{\sin \xi \sin \eta}{\xi\eta} \right|^3 d\xi d\eta \right)^{1/3}$$

$$= B\epsilon^{4/3}.$$

Finally,

$$\text{meas. } H \leq \text{meas. } E \leq \sum_{j=1}^{s} \text{meas. } E_j = Bs\epsilon^{4/3} = B[1 + (M\mu/\epsilon)]\epsilon^{4/3} = O(\epsilon^{1/3})$$

and since ϵ can be made arbitrarily small we have

$$\text{meas. } H = 0.$$

The above reasoning breaks down for $n = 2$. But in this case the proof that meas. $H = 0$ follows from quite elementary geometric considerations. I and II imply, in view of a well-known theorem, the Riemann integrability of G_a.

3. We shall base our future considerations on the following formula

$$G_a(\phi_1, \cdots, \phi_n) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \left[\int_0^{\mu} \cos(\xi F) \, |F'| \, dt \right] d\xi. \tag{6}$$

To prove (6) let the α's, $0 \leq \alpha_1 < \alpha_2 < \cdots < \alpha_r \leq \mu$, be the turning points of F so that

$$\int_0^{\mu} \cos(\xi F) \, |F'| \, dt = \sum_{j=0}^{r} \pm ([\sin \beta_{j+1}\xi - \sin \beta_j\xi]/\xi), \tag{7}$$

where $\beta_0 = F(0)$, $\beta_{r+1} = F(\mu)$, $\beta_j = F(\alpha_j)$ $(1 \leq j \leq r)$ and a $+$ or $-$ sign is assigned according as $\beta_{j+1} > \beta_j$ or $\beta_{j+1} < \beta_j$. By a well-known formula of Dirichlet

$$\left| \frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{\sin \beta_{j+1}\xi - \sin \beta_j\xi}{\xi} d\xi \right|$$

is equal to 2, 1 or 0 according as $\beta_j\beta_{j+1} < 0$, $\beta_j\beta_{j+1} = 0$ or $\beta_j\beta_{j+1} > 0$.[5] The signs in (6) are such that

$$(8)\qquad \frac{1}{2\pi}\int_{-\infty}^{+\infty}\left[\int_0^{\mu}\cos(\xi F)\mid F'\mid dt\right]d\xi = \frac{1}{2}\sum_{j=0}^{r}\left|\frac{1}{\pi}\int_{-\infty}^{+\infty}\frac{\sin\beta_{j+1}\xi - \sin\beta_j\xi}{\xi}\,d\xi\right|$$

and the right side is easily seen to be equal to the number of zeros of F in $(0,\mu)$. This proves (6). Let us notice that r (the number of turning points of F in $(0,\mu)$) although depending on $\phi_1,\cdots,\phi_n$, remains bounded. In fact r is the number of zeros of F' in $(0,\mu)$ and the reasoning given in **2** I is applicable to this case. Thus there exists a number C such that

$$(9)\qquad r < C.$$

4.[6] Let

$$G_a^{(m)}(P) = \frac{1}{2\pi}\int_{-m}^{m}\left[\int_0^{\mu}\cos(\xi F)\mid F'\mid dt\right]d\xi.$$

It is well known that for every m and b

$$\left|\frac{1}{2\pi}\int_{-m}^{m}\frac{\sin b\xi}{\xi}\,d\xi\right| < 1$$

and hence by (7) and (9) we have

$$G_a^{(m)}(P) < C.$$

Now, for every P, $G_a^{(m)}(P) \to G_a(P)$ as $m \to \infty$ and since the $G_a^{(m)}(P)$ are uniformly bounded we have by a well-known theorem that

$$(10)\qquad \lim_{m\to\infty}\int_0^1\cdots\int_0^1 G_a^{(m)}(P)\,d\phi_1\cdots d\phi_n = \int_0^1\cdots\int_0^1 G_a(P)\,d\phi_1\cdots d\phi_n.$$

On the other hand

$$\int_0^1\cdots\int_0^1 G_a^{(m)}(P)\,d\phi_1\cdots d\phi_n = \frac{1}{2\pi}\int_{-m}^{m}\left\{\int_0^1\cdots\int_0^1\left[\int_0^{\mu}\cos(\xi F)\mid F'\mid dt\right]d\phi_1\cdots d\phi_n\right\}d\xi$$

---

[5] It is obvious that $\beta_j \neq \beta_{j+1}$ and therefore β_j and β_{j+1} cannot be simultaneously 0. If 0 or μ is a root of $F = 0$ it is counted as $\frac{1}{2}$ in formula (6). Since the set of those P's for which either 0 or μ is a root, is of measure zero the following considerations are not affected by this peculiarity of formula (6).

[6] This section employs exactly the same reasoning as the one in *loc. cit.*[3]

which together with (10) implies that

$$L_a(\xi) = \int_0^1 \cdots \int_0^1 \left[\int_0^\mu \cos(\xi F) \mid F' \mid dt \right] d\phi_1 \cdots d\phi_n$$

is integrable in $(-\infty, \infty)$ and that

$$\int_0^1 \cdots \int_0^1 G_a(P) d\phi_1 \cdots d\phi_n = \frac{1}{2\pi} \int_{-\infty}^{+\infty} L_a(\xi) d\xi. \tag{11}$$

5. It remains now to compute $L_a(\xi)$. It is well known that for every real γ,

$$\mid \gamma \mid = \frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{1 - \cos \gamma\eta}{\eta^2} d\eta$$

and therefore

$$\mid F' \mid = \frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{1 - \cos(\eta F')}{\eta^2} d\eta.$$

Hence

$$L_a(\xi) = \frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{D(\xi, \eta)}{\eta^2} d\eta,$$

where

$$D(\xi, \eta) = \int_0^\mu \int_0^1 \cdots \int_0^1 [\cos(\xi F) - \cos(\eta F) \cos(\eta F')] d\phi_1 \cdots d\phi_n dt$$

and the interchange of the order of integration is justified by the fact that the only infinite integral occurring in the formula is absolutely convergent. Now,

$$\begin{aligned} \cos(\xi F) \cos(\eta F') = {} & \tfrac{1}{4} \exp\{i(\xi F + \eta F')\} + \tfrac{1}{4} \exp\{i(\xi F - \eta F')\} \\ & + \tfrac{1}{4} \exp\{i(-\xi F + \eta F')\} + \tfrac{1}{4} \exp\{i(-\xi F - \eta F')\} \end{aligned}$$

and it follows (see (5)) that

$$\begin{aligned} \int_0^1 \cdots \int_0^1 & \cos(\xi F) \cos(\eta F') d\phi_1 \cdots d\phi_n \\ & = \cos a\xi \prod_{k=1}^n J_0(\mid a_k \mid \sqrt{\xi^2 + 4\pi^2 \lambda_k^2 \eta^2}). \end{aligned}$$

Putting $\eta = 0$ we get

$$\int_0^1 \cdots \int_0^1 \cos(\xi F) d\phi_1 \cdots d\phi_n = \cos a\xi \prod_{k=1}^n J_0(\mid a_k \mid \xi).$$

Thus

$$L_a(\xi) = \frac{\mu \cos a\xi}{\pi} \int_{+\infty}^{-\infty} \frac{\prod_{k=1}^{n} J_0(|a_k|\xi) - \prod_{k=1}^{n} J_0(|a_k| \sqrt{\xi^2 + 4\pi^2\lambda_k^2\eta^2})}{\eta^2} d\eta$$

and finally by using (4) and (11)

$$E(a) = \frac{1}{2\pi^2} \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \cos a\xi \frac{\prod_{k=1}^{n} J_0(|a_k|\xi) - \prod_{k=1}^{n} J_0(|a_k| \sqrt{\xi^2 + 4\pi^2\lambda_k^2\eta^2})}{\eta^2} d\eta d\xi.$$

6. Since $J_0(|u|) = O(|u|^{-1/2})$ as $|u| \to \infty$ we see that for $n \geq 5$ the integrand of (12) is absolutely integrable. It may also be noted that if $a_1 = a_2 = \cdots a_n = 1$ a simple asymptotic formula can be obtained if the λ's satisfy certain conditions. In fact, we have in this case

$$E(a) = \frac{1}{2\pi^2} \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \cos a\xi \frac{J_0^n(\xi) - \prod_{k=1}^{n} J_0(\sqrt{\xi^2 + 4\pi^2\lambda_k^2\eta^2})}{\eta^2} d\eta d\xi$$

and letting $\xi = u/\sqrt{n}$, $\eta = v/2\pi\sqrt{n}$, $a = b\sqrt{n}$ we obtain

$$E(b\sqrt{n}) = \frac{1}{\pi} \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \cos bv \frac{J_0^n(u/\sqrt{n}) - \prod_{k=1}^{n} J_0(\sqrt{u^2 + \lambda_k^2 v^2/n})}{v^2} dvdu.$$

Assuming now that

$$\lim_{n\to\infty} n^{-1} \sum_{k=1}^{n} \lambda_k^2 = \lambda, \qquad \lim_{n\to\infty} n^{-2} \sum_{k=1}^{n} \lambda_k^4 = 0$$

we have

$$J_0^n(u/\sqrt{n}) \to \exp(-u^2/2)$$

$$\prod_{k=1}^{n} J_0(\sqrt{u^2 + \lambda_k^2 v^2/n}) \to \exp(-u^2/2) \exp(-\lambda v^2/2)$$

and the convergence is uniform in every bounded region of the (u, v) plane. Thus, since all the integrals involved are absolutely convergent,

$$E(b\sqrt{n}) \sim \frac{1}{\pi} \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \cos bu \exp(-u^2/2) \frac{1 - \exp(-\lambda v^2/2)}{v^2} dvdu.$$

The last integral can be easily evaluated and we obtain the formula

$$E(b\sqrt{n}) \sim \frac{2}{\pi} \sqrt{\lambda} \exp(-b^2/2).$$

CORNELL UNIVERSITY,
ITHACA, N. Y.

ON DOUBLE STURM-LIOUVILLE SERIES.*

By JOSEPHINE MITCHELL.

Introduction. In this paper we consider the *double Sturm-Liouville series* (SL series) of a function, Lebesgue integrable [1] on the square Q ($0 \leqq x \leqq \pi$, $0 \leqq y \leqq \pi$), that is, the Fourier development of this function with respect to the double orthonormal system, $\{\phi_m(x)\phi_n(y)\}$, $(m, n = 0, 1, 2, \cdots)$ (1). Our main method is the comparison of the double SL series of a given function $f(x, y)$ with its Fourier cosine-cosine series (FCC series). A. Haar [9] introduced this idea in the one-dimensional case, proving that the difference of the partial sums of the SL and Fourier cosine developments of an integrable function $f(x)$ converges to 0 uniformly on $(0, \pi)$; briefly, the two series are said to be uniformly *equiconvergent*.[2] The proof depends mainly upon the fact that the difference between the SL and cosine kernels (the *difference kernel*) is bounded [9].

It is natural to generalize the concepts of "equiconvergence" and "equisummability" to the double SL and FCC series. However the "cross product" terms of the corresponding difference kernels are *not bounded* (as Haar has pointed out), so that in the case of equiconvergence we have found it necessary to put some restriction on the given function to be expanded.

In discussing the *equiconvergence* of the double SL and FCC series (Part III) we assume that the given function is of bounded variation (Tonelli) ([17], p. 443) and obtain *equiconvergence of the SL and FCC series over the square Q* (Theorem III). Since this condition is included in all of the older sufficient conditions for the convergence of the double Fourier series as well as in some of the more modern ones,[3] we get various *sufficient conditions for the convergence of the double SL series* (Theorem IV).

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[1] In this paper integration is Lebesgue integration. Also, a measurable function f is said to be of *class* L^p if $|f|^p$ is integrable.

[2] We may consider instead of convergence some of the various methods of summation of series and define *equisummability* of two series.

[3] Well-known sufficient conditions for the convergence of the double Fourier series are, e. g., that the function be of bounded variation (Hardy) [10] or of bounded variation (Arzela) [7]. Other sufficient conditions are given by L. Tonelli [17] and in generalizations to two variables of the tests of Dini, de la Vallée Poussin, Lebesgue, etc. For a statement of the tests and the connections among them see J. J. Gergen's paper [8].

Part II contains our principal theorem (Theorem I) on the (C, 1, 1) *equisummability of the double SL and FCC series,* namely that the *SL series of an integrable function* $f(x, y)$ *is* $(C, 1, 1)$ *equisummable with its FCC series at every point* (x, y) *in* Q *for which*

$$\text{(A)} \quad \frac{1}{hk}\int_0^h \int_0^k |f(x+u, y+v)|\, du\, dv \leqq C(x, y) < \infty, \quad (0 < |h| \leqq \pi, 0 < |k| \leqq \pi).$$

We also prove that the two series are *equisummable almost everywhere* (a. e.) *in* Q (Theorem II).

To get a *summability* theorem for double SL series we consider the corresponding theory for the summability of the double FCC series. In 1918 H. Geiringer ([7], also [17]) following the method of proof introduced by Lebesgue, proved that the Fourier series of $f(x, y)$ is (C, 1, 1) summable to $f(x, y)$ at every point (x, y) where

$$\text{(C)} \quad \lim_{h\to 0,\, k\to 0} \frac{1}{hk} \int_0^h \int_0^k |f(x+u, y+v) - f(x, y)|\, du\, dv = 0,$$

if, also, the two simple integrals of $|f|$ are bounded. A. Zygmund [19] proved the summability at all points (x, y) for which (C) holds and

$$\text{(D)} \quad \frac{1}{hk} \int_0^h \int_0^k |f(x+u, y+v) - f(x, y)|\, du\, dv < D(x, y) < \infty, \quad (0 < |h| \leqq \pi,\ 0 < |k| \leqq \pi).$$

Relating these conditions to the problem of "strong derivability" of double integrals, which has been thoroughly investigated in the last few years by S. Saks, A. Zygmund and others [19], he showed that (C) and (D) hold a. e. for a function of class L^p $(p > 1)$ and that therefore the Fourier series of such a function is summable a. e. A more recent investigation by B. Jessen, J. Marcinkiewicz and A. Zygmund [16] has extended this result to a class of functions $f(x, y)$ such that $f \log^+ |f|$ (cf. footnote [11]) is integrable (denoted by class (B)). By showing that for such functions condition (A) holds a. e.[4] we deduce that the *SL series of a function* $f(x, y)$ *of class* (B) *is* (C, 1, 1) *summable to* $f(x, y)$ a. e.

Finally in part IV we consider a *modification of the Abel-Poisson summation method.* Instead of multiplying the n-th term of the Fourier development, $\sum_{n=0}^{\infty} c_n v_n(\theta)$, of a given function $f(\theta)$, defined on $(-\pi, \pi)$, by r^n, we form the sequence $\{v_n(\theta, r)\}$, $(n = 0, 1, 2, \cdots)$, where $v_n(\theta, r)$ is *har-*

[4] For a discussion of the relation between functions of class (B) and conditions (C), (D) and (A) see 4.

monic in the interior of the unit circle and reduces to $v_n(\theta)$ *on the circumference and consider the limit of the series* $\sum_{n=0}^{\infty} c_n v_n(\theta, r)$ *as* r *approaches* $1-0$ (*modified Poisson method of summability*) (**6.1**). For ordinary Fourier series this reduces to the usual Abel-Poisson summability. This procedure, which can be generalized in various ways to more variables, was introduced by S. Bergman ([4], [5]) in a much more general connection in the theory of functions of two complex variables. Applying this summation to the SL system we show that the *double SL series of an integrable function is equisummable with its FCC series* (Theorem V). Lastly we state a theorem on modified Poisson summability for the development of an integrable function with respect to more general orthonormal systems (Theorem VI).

PART I. The Sturm-Liouville System

1. Definitions and theorems.

1. The SL system ([9], [18]). Consider the *SL orthonormal system* $\{\phi_n(x)\}$, $(n = 0, 1, 2, \cdots)$, defined on $(0 \leqq x \leqq \pi)$. The $\phi_n(x)$ are the characteristic functions of the *differential system*

$$(1) \quad d^2u/dx^2 + (Q(x) + \lambda)u = 0$$

$$du/dx - hu = 0 \text{ for } x = 0,\ du/dx + Hu = 0 \text{ for } x = \pi (h, H \text{ constants}),$$

where λ is a parameter $(-\infty < \lambda < \infty)$ and $Q(x)$ is continuous. If, also, $Q(x)$ is of bounded variation, Liouville, and Hobson [12], have proved the *asymptotic formula*

$$(2) \quad \phi_n(x) = (2/\pi)^{1/2} \cos nx + \beta(x)\frac{\sin nx}{n} + \frac{\rho_n(x)}{(n+1)^2}, \quad (n = 0, 1, 2, \cdots),$$

where the function $\beta(x)$ has a continuous derivative, the continuous functions $\rho_n(x)$ are bounded in absolute value independently of n and x, and for $n = 0$ the second term on the right is zero.

Now let $f(x)$ be a function, integrable on $(0, \pi)$. The development of $f(x)$ by the SL system

$$(3) \qquad \sum_{n=0}^{\infty} a_n \phi_n(x), \qquad a_n = \int_0^{\pi} f(s)\phi_n(s)\,ds, \qquad (n = 0, 1, 2, \cdots),$$

is called the *SL series* of $f(x)$.

It has been proved that the SL system is *complete with respect to integrable functions* [18].

Using the system $\{\phi_m(x)\}$, we can form the *double SL orthonormal system* $\{\phi_m(x)\phi_n(y)\}$, $(m, n = 0, 1, 2, \cdots)$, where (x, y) is a point in $Q(0 \leqq x \leqq \pi,\ 0 \leqq y \leqq \pi)$.

Now let $f(x, y)$ be an integrable function in Q.[5] Its *double SL series* is

$$\sum_{m=0,\ n=0}^{\infty} a_{mn}\phi_m(x)\phi_n(y) \tag{4}$$ [6]

where

$$a_{mn} = \int_0^{\pi}\int_0^{\pi} f(s, t)\phi_m(s)\phi_n(t)\,ds\,dt, \qquad (m, n = 0, 1, 2, \cdots). \tag{5}$$

In this paper we compare the *double SL series of* $f(x, y)$ *with its double FCC series*

$$\sum_{m=0,\ n=0}^{\infty} \lambda_{mn} b_{mn} \cos mx \cos ny \tag{6}$$

$\lambda_{mn} = \frac{1}{4}$ if $m = n = 0$, $= \frac{1}{2}$ if $m = 0$, $n > 0$ or $m > 0$, $n = 0$, $= 1$ if $m > 0$, $n > 0$ where

$$b_{mn} = \frac{4}{\pi^2}\int_0^{\pi}\int_0^{\pi} f(s, t)\cos ms \cos nt\,dsdt, \qquad (m, n = 0, 1, 2, \cdots). \tag{7}$$

2. Equiconvergence and (C, 1) equisummability for $f(x)$.

Denote the *partial sums of the SL and cosine kernels,* respectively, by

$$K_n(x, s) = \sum_{\nu=0}^{n} \phi_\nu(x)\phi_\nu(s), \tag{8}$$

and

$$C_n(x, s) = (2/\pi)\left(\tfrac{1}{2} + \sum_{\nu=1}^{n} \cos \nu x \cos \nu s\right), \tag{9}$$

and define the "*difference kernel,*" $\Phi_n(x, s)$, by

$$\Phi_n(x, s) = K_n(x, s) - C_n(x, s). \tag{10}$$

Similarly denote the corresponding *arithmetic means* ((C, 1) means) by

$$K^*_n(x, s) = \frac{K_0(x, s) + \cdots + K_{n-1}(x, s)}{n} \tag{11}$$

and

$$C^*_n(x, s) = \frac{C_0(x, s) + \cdots + C_{n-1}(x, s)}{n} \tag{12}$$

and define the "(C, 1) *difference kernel,*" $\Phi^*_n(x, s)$, as in (10).

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[5] Whenever we consider $f(x, y)$ beyond the square Q, we extend it as an even-even function of period 2π in each of the variables x and y.

[6] The *completeness of the double SL system* with respect to integrable functions is an immediate consequence of the corresponding property for the simple system ([20] p. 13, Ex. 6).

Haar's proof of the equiconvergence theorem and the analogous proof for the equisummability theorem involve the *boundedness of the difference kernel* [9], that is,

$$|\Phi_n(x,s)| \leqq \Phi, \tag{13}$$

where Φ is independent of n, x and s, and

$$|\Phi^*_n(x,s)| = \frac{|\Phi_0(x,s) + \cdots + \Phi_{n-1}(x,s)|}{n} \leqq \Phi. \tag{14}$$

THEOREM 1[7] (Haar). *The SL series of a function $f(x)$, integrable on $(0,\pi)$, is uniformly equiconvergent with its Fourier cosine (FC) series* [9].

THEOREM 2. *The SL series of a function $f(x)$, integrable on $(0,\pi)$, is uniformly equisummable $(C,1)$ with its FC series.*

2. Two properties of the difference kernels $\Phi_n(x,s)$, $\Phi^*_n(x,s)$. The following lemmas enable us to prove general $(C,1,1)$ equisummability theorems for the double SL and FCC series[8] (cf. Part II).

LEMMA I. *The difference kernel $\Phi_n(x,s)$ converges to 0 uniformly with respect to s as $n \to \infty$ if s is in any closed interval not containing the point x.*

Proof. Let x be an arbitrary fixed point in $(0,\pi)$.

From the asymptotic formula 1 (2) for the SL functions we easily see that the sequence $\Phi_n(x,s)$ converges uniformly for s in any closed interval (a,b) in $(0,\pi)$, not containing the point x, as $n \to \infty$. Hence the limit $\Phi(x,s)$ is a continuous function of s in (a,b).

Further $|\Phi_n(x,s)| \leqq \Phi$ so that

$$\lim_{n\to\infty} \int_a^b \Phi_n(x,s)\,ds = \int_a^b \Phi(x,s)\,ds. \tag{1}$$

But from Haar's theorem

$$\lim_{n\to\infty} \int_a^b \Phi_n(x,s)\,ds = 0, \tag{2}$$

for $\int_a^b K_n(x,s)\,ds$ is the partial sum of the SL development of the function equal to 1 in (a,b) and 0 elsewhere, and similarly for $\int_a^b C_n(x,s)\,ds$.

Consequently $\Phi(x,s) = 0$ for all s in $(0,\pi)$ different from x.

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[7] We indicate known theorems and lemmas by arabic numbers and new ones by Roman numerals.

[8] We intend to use Lemmas I and II again in a later discussion on the equiconvergence of the SL and FCC series.

This *same property* is, of course, valid for the kernel $\Phi^*_n(x,s)$ defined in 1.[9]

LEMMA II.

$$\lim_{n\to\infty}\int_0^\pi |\Phi_n(x,s)|\,ds = 0 \text{ for all } x \text{ in } (0,\pi). \tag{3}$$

Proof. This property follows for each x in $(0,\pi)$ from Lemma I by the Lebesgue theorem on the integration of sequences. The same property holds for the kernel $\Phi^*_n(x,s)$.

COROLLARY. *If $f(x)$ is integrable over $(0,\pi)$, then for all x in $(0,\pi)$*

$$\lim_{n\to\infty}\int_0^\pi |\Phi_n(x,s)|\,f(s)\,ds = 0. \tag{4}$$

PART II. On the (C, 1, 1) Equisummability of the double SL and double FCC Series.

3. Theorem on Equisummability.

1. THEOREM I. *Let $f(x,y)$ be integrable in the square Q $(0\leqq x\leqq\pi,\ 0\leqq y\leqq\pi)$. Then the double SL series of $f(x,y)$ is (C, 1, 1) equisummable with its double FCC series at every point (x,y) in Q for which*

$$\text{(A)}\quad \frac{1}{hk}\int_0^h\int_0^k |f(x+u,y+v)|\,du\,dv \leqq C,\ (0<|h|\leqq\pi,\ 0<|k|\leqq\pi),$$

where $C = C(x,y) < \infty$.[10]

Proof. Denote the SL and FCC (C, 1, 1) partial sums, respectively, by

$$\sigma_{mn}(x,y) = \int_0^\pi\int_0^\pi K^*_m(x,s)K^*_n(y,t)f(s,t)\,ds\,dt \tag{1}$$

$$\sigma'_{mn}(x,y) = \int_0^\pi\int_0^\pi C^*_m(x,s)C^*_n(y,t)f(s,t)\,ds\,dt. \tag{2}$$

Then $\sigma_{mn}(x,y) - \sigma'_{mn}(x,y)$ equals the sum of the three integrals

$$\Psi^{(1)}_{mn}(x,y) = \int_0^\pi\int_0^\pi \Phi^*_m(x,s)\Phi^*_n(y,t)f(s,t)\,ds\,dt, \tag{3}$$

$$\Psi^{(2)}_{mn}(x,y) = \int_0^\pi\int_0^\pi C^*_m(x,s)\Phi^*_n(y,t)f(s,t)\,ds\,dt, \tag{4}$$

$$\Psi^{(3)}_{mn}(x,y) = \int_0^\pi\int_0^\pi \Phi^*_m(x,s)C^*_n(y,t)f(s,t)\,ds\,dt. \tag{5}$$

---

[9] It is well-known that Lemma I is valid for the Fejér kernel $C^*_n(x,s)$. Thence *it also holds for the SL kernel $K^*_n(x,s)$*. However we do not use this fact.

[10] For a further discussion of condition (A) see 4. 2.

It will, obviously, be sufficient to prove that $|\Psi^{(1)}_{mn}|$, $|\Psi^{(2)}_{mn}|$ and $|\Psi^{(3)}_{mn}| \to 0$ as $m, n \to \infty$.

2. The fact that $|\Psi^{(1)}_{mn}(x,y)| \to 0$ follows at once from the uniform boundednesss of Φ^*_m.

Now

$$C^*_m(x,s) = \frac{1}{m\pi}\left\{\frac{\sin^2\frac{1}{2}m(s-x)}{2\sin^2\frac{1}{2}(s-x)} + \frac{\sin^2\frac{1}{2}m(s+x)}{2\sin^2\frac{1}{2}(s+x)}\right\} \tag{6}$$

so that in discussing $\Psi^{(2)}_{mn}(x,y)$ we consider the two integrals

$$\Psi'_{mn}(x,y) = \frac{1}{m\pi}\int_0^\pi\int_0^\pi \frac{\sin^2\frac{1}{2}m(s+x)}{2\sin^2\frac{1}{2}(s+x)}\,\Phi^*_n(y,t)f(s,t)\,ds\,dt \tag{7}$$

and

$$\Psi''_{mn}(x,y) = \frac{1}{m\pi}\int_0^\pi\int_0^\pi \frac{\sin^2\frac{1}{2}m(s-x)}{2\sin^2\frac{1}{2}(s-x)}\,\Phi^*_n(y,t)f(s,t)\,ds\,dt. \tag{8}$$

If $x = 0$ or π, $\Psi'_{mn}(x,y)$ reduces to the same value as $\Psi''_{mn}(x,y)$. If x is an interior point, we can choose a $\delta > 0$ so that $0 \leqq x-\delta < x+\delta \leqq \pi$ and then

$$|\Psi'_{mn}(x,y)| \leqq \frac{1}{m\pi}\,\frac{1}{2\sin^2\frac{1}{2}\delta}\,\Phi\int_0^\pi\int_0^\pi |f(s,t)|\,ds\,dt. \tag{9}$$

For fixed δ, a sufficiently large m gives

$$|\Psi'_{mn}(x,y)| < \eta, \text{ where } \eta \text{ is an arbitrary positive number.} \tag{10}$$

To consider the integral $\Psi''_{mn}(x,y)$ we decompose it into

$$\int_0^\pi\int_0^\pi = \left\{\int_0^x + \int_x^\pi\right\}\left\{\int_0^{y-\epsilon} + \int_{y-\epsilon}^{y} + \int_y^{y+\epsilon} + \int_{y+\epsilon}^{\pi}\right\}. \tag{11}$$

Let

$$g(h,k) = \int_0^k |f(x+h, y+v)|\,dv, \theta(h,k) = \int_0^h\int_0^k |f(x+u, y+v)|\,du\,dv. \tag{12}$$

It is readily shown that

$$\frac{1}{m\pi}\,\frac{\sin^2\frac{1}{2}mu}{\sin^2\frac{1}{2}u} \leqq \frac{\pi m}{(1+\frac{1}{2}mu)^2} \quad \text{for } 0 < u \leqq \pi \text{ (e. g. [17], p. 176).} \tag{13}$$

Then

$$\begin{aligned}
|I_1| &= \left|\frac{1}{m\pi}\int_x^\pi\int_y^{y+\epsilon}\frac{\sin^2\frac{1}{2}m(s-x)}{2\sin^2\frac{1}{2}(s-x)}\,\Phi^*_n(y,t)f(s,t)\,ds\,dt\right| \qquad (14)\\
&\leqq \frac{1}{m\pi}\,\Phi\int_0^{\pi-x}\int_0^{\epsilon}\frac{\sin^2\frac{1}{2}mu}{2\sin^2\frac{1}{2}u}\,|f(x+u, y+v)|\,du\,dv\\
&\leqq \tfrac{1}{2}\pi\Phi\int_0^{\pi-x}\frac{m}{(1+\frac{1}{2}mu)^2}\,g(u,\epsilon)\,du\\
&\leqq \tfrac{1}{2}\pi\Phi\int_0^{\pi}\frac{m}{(1+\frac{1}{2}mu)^2}\,g(u,\epsilon)\,du.
\end{aligned}$$

Integration by parts of the right hand integral with condition (A) gives us

$$(15)\qquad |I_1| \leqq \tfrac{1}{2}\pi\Phi \left\{ \frac{m\theta(\pi,\epsilon)}{(1+\frac{1}{2}m\pi)^2} + \int_0^\pi \frac{m^2\theta(u,\epsilon)}{(1+\frac{1}{2}mu)^3}du \right\}$$

$$\leqq \tfrac{1}{2}\pi\Phi \left\{ \frac{mC\pi\epsilon}{(1+\frac{1}{2}m\pi)^2} + C\epsilon\int_0^\pi \frac{m^2u}{(1+\frac{1}{2}mu)^3}du \right\}$$

$$\leqq \tfrac{1}{2}\pi\Phi \left\{ \frac{mC\pi\epsilon}{(1+\frac{1}{2}m\pi)^2} + 2C\epsilon \right\}.$$

Secondly let

$$(16)\qquad M_n = M_n(\epsilon) = \max_{y+\epsilon\leqq t\leqq\pi} \Phi^*{}_n(y,t).$$

By Lemma I, $M_n \to 0$ as $n \to \infty$ for fixed ϵ. Then

$$(17)\qquad |I_2| = \left| \frac{1}{m\pi}\int_x^\pi\int_{y+\epsilon}^\pi \frac{\sin^2\frac{1}{2}m(s-x)}{2\sin^2\frac{1}{2}(s-x)}\Phi^*{}_n(y,t)f(s,t)\,ds\,dt \right|$$

$$\leqq \tfrac{1}{2}\pi M_n \int_0^\pi \frac{m}{(1+\frac{1}{2}mu)^2}g(u,\pi)\,du.$$

Again integration by parts with condition (A) gives

$$(18)\qquad |I_2| \leqq \tfrac{1}{2}\pi M_n \left\{ \frac{mC\pi^2}{(1+\frac{1}{2}m\pi)^2} + 2C\pi \right\}.$$

Now it is readily seen that Ψ''_{mn} decomposed as in (11), involves four integrals of type I_1 and four of type I_2, satisfying inequalities like (15) and (18) respectively. From inequalities (15) and (18) and Lemma I, it now follows at once that $\lim_{m,n} \Psi''_{mn}(x,y) = 0$. Consequently

$$(19)\qquad \lim_{m\to\infty,\ n\to\infty} |\Psi^{(2)}_{mn}(x,y)| = 0.$$

Similarly for $|\Psi^{(3)}_{mn}(x,y)|$ and the theorem is proved.

4. Summability for the double SL series and further results on Equi-summability.

1. It is easily deduced that condition (A) is satisfied at almost every point (x,y) provided that

(B) $f(x,y)\log^+|f(x,y)|$ belongs to L over Q [11] ([14], p. 221).

On the other hand it is well-known that the Fourier series of functions satisfying condition (B) are summable (C, 1, 1) to $f(x,y)$ almost everywhere [14].[12] It follows that the *double SL series of the function f satisfying* (B) *is summable* (C, 1, 1) *to* $f(x,y)$ *a. e.*

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[11] $\mathrm{Log}^+|f(x,y)|$ is defined as equal to $\log|f(x,y)|$ if $|f(x,y)|>1$, equal to 0 otherwise (cf. e. g. [20], p. 150). $f\log^+|f|$ of class L implies f of class L ([11], p. 99).

[12] If $f(x,y)$ is extended beyond Q as an even-even function its double Fourier series reduces to the double FCC series.

We shall prove now the following

THEOREM II. *Let $f(x,y)$ be integrable over the square Q. Then the double SL series of $f(x,y)$ is $(C,1,1)$ equisummable with its double FCC series at almost every point (x,y) in Q.*[13]

Proof. It is sufficient to prove that $\Psi^{(2)}_{mn}(x,y) \to 0$ a. e. We use the fact that $\int_0^\pi \int_0^\pi C^*_m(x,s) f(s,t)\,ds\,dt$ converges to $\int_0^\pi f(x,t)\,dt$ at all points x where the derivative of $\int_0^h \int_0^\pi |f(x+u,t) - f(x,t)|\,du\,dt$ exists and equals 0 (a. e.).[14]

Now

$$|\Psi^{(2)}_{mn}(x,y)| \leqq |J_1| + |J_2| \tag{1}$$

where

$$J_1 = \int_0^\pi \int_0^\pi C^*_m(x,s)\Phi^*_n(y,t) f(x,t)\,ds\,dt \tag{2}$$

$$= \int_0^\pi \Phi^*_n(y,t) f(x,t)\,dt$$

and

$$J_2 = \int_0^\pi \int_0^\pi C^*_m(x,s)\Phi^*_n(y,t)\,(f(s,t) - f(x,t))\,ds\,dt. \tag{3}$$

By Fubini's theorem $f(x,t)$ is integrable with respect to t for a. a. x. Therefore by Theorem 2 the right side of (2) $\to 0$ as $n \to \infty$ for a. a. x and uniformly in y. Hence $J_1 \to 0$ as $n \to \infty$ for a. a. (x,y) in Q.

Again

$$|J_2| \leqq \Phi \int_0^\pi C^*_m(x,s)\,g(s)\,ds \tag{4}$$

where $g(s) = \int_0^\pi |f(s,t) - f(x,t)|\,dt$. From the theory for simple Fourier series the second factor on the right of (4) $\to 0$, as $m \to \infty$, at all points x for which the derivative of the indefinite integral of $g(s)$ is 0 (a. a. x). Hence $J_2 \to 0$ as $m \to \infty$ for a. a. (x,y) in Q. Consequently $|\Psi^{(2)}| \to 0$ as $m, n \to \infty$. Similarly for $|\Psi^{(3)}|$ and the theorem is proved.[15]

2. It has been proved that the double FCC series of an integrable function $f(x,y)$ is $(C,1,1)$ summable to $f(x,y)$ at all points for which the two

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[13] This theorem was suggested by the referee.

[14] The latter statement can be proved by a generalization of the proof of the corresponding one-variable theorem given in ([13], vol. 1, p. 582).

[15] It must be noted that this proof of the fact that $|\Psi^{(2)}_{mn}| \to 0$ does not use Lemma I. However the proof seems to us less straight-forward than that of Theorem I.

conditions (C) and (D) (see Introduction) hold and that for functions of class L^p $(p > 1)$ these conditions hold a. e. [19]. We now point out that even for *functions of class* (B) *condition* (C) *holds a. e.* and that moreover (C) *implies* (D) *a. e.*, thus affording a second proof of the summability theorem for such functions; since obviously (D) *implies* (A) *everywhere* we have a proof of the corresponding summability theorem for double SL series which shows the connection of these different conditions.

This fact follows for *condition* (C) from the recently proved theorem that the *integral of a function f of class* (B) *is "strongly differentiable"* [16] *to* $f(x, y)$ *a. e.*[17] by Lebesgue's argument ([13] vol. 1, p. 582).

Condition (D) can be proved to result from (C) a. e. as follows. If h, k are sufficiently small $(< \delta_1)$ (D) is given by the statement of condition (C). Also (D) follows readily if $h, k \geqq \delta_1$. If $h < \delta_1$ and $k \geqq \delta_1$, then

$$\frac{1}{hk}\int_0^h\int_0^k |f(x+u, y+v) - f(x, y)|\, du dv \leqq \frac{1}{h\delta_1}\int_0^h \left(\int_0^\pi |f(x+u, t)|\, dt\right) du + |f(x, y)|. \tag{5}$$

Now the monotone function $\int_0^x \left(\int_0^\pi |f(s, t)|\, dt\right) ds$ is differentiable to $\int_0^\pi |f(x, t)|\, dt$ at a. e. point x in $(0, \pi)$ (even if $f(s, t)$ belongs to L). Consequently for such x and sufficiently small h $(h < \delta_2)$

$$\frac{1}{h}\int_0^h\int_0^\pi |f(x+u, t)|\, du\, dt \leqq \int_0^\pi |f(x, t)|\, dt + \epsilon. \tag{6}$$

Hence

$$\frac{1}{hk}\int_0^h\int_0^k |f(x+u, y+v) - f(x, y)|\, du\, dv \leqq \frac{1}{\delta_1}\left(\int_0^\pi |f(x, t)|\, dt + \epsilon\right) + |f(x, y)|. \tag{7}$$

A similar result holds if $h \geqq \delta_1$ and $k < \delta_1$. Therefore from the results of the four cases condition (D) follows from (C) a. e.

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[16] *Strong derivability* of a double integral means that

$$(1/hk)\int_0^h\int_0^k f(x+u, y+v)\, du dv \to f(x, y) \quad \text{as} \quad h, k \to 0.$$

[17] Such theorems have been proved by S. Saks, B. Jessen, J. Marcinkiewicz and A. Zygmund, etc. For the particular statement of the theorem used here see ([16], p. 122).

PART III. On the uniform Equiconvergence of the double SL and FCC Series.

5. Equiconvergence theorem and sufficient conditions for the convergence of the double SL series.

1. In this section we proye a theorem on the equiconvergence of the double SL and FCC series of an integrable function $f(x, y)$. Since the difference kernel is not bounded a proof similar to that of Haar's theorem cannot be generalized to two variables. Further, the Fourier kernel is not positive as in the case for Cesàro summability. We prove equiconvergence for integrable functions $f(x, y)$ which are of *bounded variation* as defined by *Tonelli* (b. v. T.). This condition is only slightly stronger than that of bounded variation in each variable and is by no means sufficient for the convergence of the double Fourier series as L. Tonelli has shown by a counter-example ([17], p. 485).

DEFINITION. *A function $f(x, y)$ is said to be of bounded variation (Tonelli) in Q if*

(i) *$f(x, y)$ is of bounded variation in $(0, \pi)$ as a function of y for almost all x,*

(ii) *$f(x, y)$ is of bounded variation in $(0, \pi)$ as a function of x for a. a. y,*

(iii) *the total variation of $f(x, y)$ with respect to y, namely,*

$$(1)\quad V_1(x) = \sup_{\sigma} \sum_{\nu=1}^{k} |f(x, y_\nu) - f(x, y_{\nu-1})|,\ \sigma;\ 0 = y_0 < y_1 < \cdots < y_k = \pi,$$

is dominated by an integrable function $U_1(x)$, that is,

$$(2)\quad V_1(x) \leqq U_1(x),$$

(iv) and similarly *the total variation of $f(x, y)$ with respect to x, namely, $V_2(y)$, is dominated by an integrable function $U_2(y)$* ([17], p. 443).[18]

2. THEOREM III. *If the integrable function $f(x, y)$ is of b. v. T. in square Q, then its double SL series is equiconvergent in Q with its double FCC series.*

Proof.[19] In analogy to the notation of **3.** 1 call the partial sums of the double SL and FCC series

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[18] L. Tonelli imposed originally integrability on $V_1(x)$ and $V_2(y)$. The extension considered here is due to C. R. Adams and J. A. Clarkson [2] and J. J. Gergen [8].

[19] The proof of this theorem suggested itself by reason of certain similar ideas used by L. Tonelli in *Serie Trigonometriche* ([17], Ch. IX).

$$S_{mn}(x,y) = \int_0^\pi \int_0^\pi K_m(x,s)K_n(y,t)f(s,t)\,ds\,dt, \tag{3}$$

$$s_{mn}(x,y) = \int_0^\pi \int_0^\pi C_m(x,s)C_n(y,t)f(s,t)\,ds\,dt, \tag{4}$$

respectively. Also, using as before $\Phi_m(x,s) = K_m(x,s) - C_m(x,s)$, we set

$$\Theta^{(1)}_{mn}(x,y) = \int_0^\pi \int_0^\pi \Phi_m(x,s)\Phi_n(y,t)f(s,t)\,ds\,dt, \tag{5}$$

$$\Theta^{(2)}_{mn}(x,y) = \int_0^\pi \int_0^\pi C_m(x,s)\Phi_n(y,t)f(s,t)\,ds\,dt, \tag{6}$$

$$\Theta^{(3)}_{mn}(x,y) = \int_0^\pi \int_0^\pi \Phi_m(x,s)C_n(y,t)f(s,t)\,ds\,dt, \tag{7}$$

and show that the three integrals $|\Theta^{(1)}_{mn}(x,y)|$, $|\Theta^{(2)}_{mn}(x,y)|$ and $|\Theta^{(3)}_{mn}(x,y)| \to 0$ as $m, n \to \infty$.

As for integral (3) **3** the convergence of $\Theta^{(1)}_{mn}(x,y)$ to 0 follows from the uniform boundedness of $\Phi_m(x,s)$.[20]

To consider $\Theta^{(2)}_{mn}(x,y) = \int_0^\pi \int_0^\pi C_m(x,s)\Phi_n(y,t)f(s,t)\,ds\,dt$ put

$$F_n(s) = F_n(s,y) = \int_0^\pi \Phi_n(y,t)f(s,t)\,dt. \tag{8}$$

Then $F_n(s)$ *is of bounded variation with respect to* s *in* $(0,\pi)$ *for all* y *in* $(0,\pi)$ *and* $(n = 0, 1, 2, \cdots)$.

Proof. Let σ be any subdivision of $(0,\pi)$, $\sigma; 0 = s_0 < s_1 < \cdots < s_k = \pi$. Then

$$\sum_{\nu=1}^{k} |F_n(s_\nu) - F_n(s_{\nu-1})| = \sum_{\nu=1}^{k} \left| \int_0^\pi \Phi_n(y,t)\,(f(s_\nu,t) - f(s_{\nu-1},t))\,dt \right| \tag{9}$$

$$\leqq \int_0^\pi |\Phi_n(y,t)| \sum_{\nu=1}^{k} |f(s_\nu,t) - f(s_{\nu-1},t)|\,dt.$$

By (1) and (2) in the definition of b. v. T

$$\sum_{\nu=1}^{k} |f(s_\nu,t) - f(s_{\nu-1},t)| \leqq V_2(t) \text{ for a. a. } t \text{ and } V_2(t) \leqq U_2(t)$$

(for all t) so that

$$\sum_{\nu=1}^{k} |F_n(s_\nu) - F_n(s_{\nu-1})| \leqq \int_0^\pi |\Phi_n(y,t)|\,U_2(t)\,dt. \tag{10}$$

[20] By applying a well-known convergence theorem due to E. W. Hobson ([13], vol. 2, pp. 422-4) we see that the convergence of $\Theta_{mn}{}^{(1)}(x,y)$ to 0 is uniform. (See 7.2).

Therefore

$$(11)\quad V_n(\pi) = \sup_{\sigma} \sum_{\nu=1}^{k} |\, F_n(s_\nu) - F_n(s_{\nu-1})\,| \leqq \int_0^\pi |\, \Phi_n(y,t)\,|\, U_2(t)\,dt.$$

Hence $F_n(s)$ [21] is of bounded variation and can be represented as $F_n(s) = F_n(0) + P_n(s) - N_n(s)$, where the positive, monotonic non-decreasing functions, $P_n(s)$ and $N_n(s)$, are the positive and negative variations, respectively, of $F_n(s)$ in $(0, s)$.

Then

$$(12)\qquad \Theta^{(2)}_{mn}(x,y) = \int_0^\pi C_m(x,s)\,(F_n(0) + P_n(s) - N_n(s))\,ds$$

and we can use the second mean value theorem getting

$$(13)\quad \Theta^{(2)}_{mn}(x,y) = F_n(0)\int_0^\pi C_m(x,s)\,ds + P_n(\pi-0)\int_\xi^\pi C_m(x,s)\,ds - N_n(\pi-0)\int_\eta^\pi C_m(x,s)\,ds,\quad (0 \leqq \xi \leqq \pi,\ 0 \leqq \eta \leqq \pi)$$

([13], vol. 1, p. 568).

Now it is well-known that $|\int_a^b C_m(x,s)\,ds| \leqq M$, where M is independent of m and x for all a, b such that $0 \leqq a \leqq b \leqq \pi$ ([13], vol. 2, p. 510). Hence

$$(14)\quad |\,\Theta^{(2)}_{mn}(x,y)\,| \leqq M(|\,F_n(0)\,| + P_n(\pi-0) + N_n(\pi-0)) = M(|\,F_n(0)\,| + V_n(\pi-0)).$$

Now $|\,F_n(0)\,| \leqq |\,F_n(s_0)\,| + V_n(\pi)$ where s_0 is such that $f(s_0, t)$ is an integrable function of t (a. a. s_0). Since also $V_n(\pi-0) \leqq V_n(\pi)$, then from inequalities (14) and (11)

$$(15)\quad |\,\Theta^{(2)}_{mn}(x,y)\,| \leqq M(|\,F_n(s_0)\,| + 2V_n(\pi)) \leqq M(|\int_0^\pi \Phi_n(y,t) f(s_0,t)\,dt\,| + 2\int_0^\pi |\,\Phi_n(y,t)\,|\,U_2(t)\,dt).$$

By Haar's theorem and the corollary to Lemma II the right side $\to 0$ as $n \to \infty$. Hence $|\,\Theta^{(2)}_{mn}(x,y)\,| \to 0$ as $m, n \to \infty$ for all (x, y) in Q. Similarly for $|\,\Theta^{(3)}_{mn}\,|$ and the theorem is proved.

3. Remark on equiconvergence by rows and columns. A double series $\sum_{m=0,\,n=0}^{\infty} A_{mn}$ is said to be *convergent by rows* if all the series $\sum_{n=0}^{\infty} A_{mn}$

[21] $F_n(s)$ may be defined by (8) for only a. a. s. For all exceptional s it may be defined as equal to either $F_n(s+0)$ or $F_n(s-0)$.

$(m = 0, 1, 2, \cdots)$ and the series $\sum_{n=0}^{\infty} (\sum_{n=0}^{\infty} A_{mn})$ converge. (Similarly for convergence by columns).

By a method similar to that used in the proof of Theorem III the following equiconvergence theorem can be proved.

THEOREM III'. *If the integrable function $f(x, y)$ is of b. v. T in Q, then its double SL series is equisummable by rows and columns with its double FCC series.*

4. Sufficient conditions for convergence of the double SL series. Any one of the following conditions on the function $f(x, y)$ is sufficient for the convergence of its double FCC series and also implies that the function is of b. v. T. [1] so that Theorem III is applicable.

(i) $f(x, y)$ is of bounded variation (Hardy) ([10], [1]),

(ii) $f(x, y)$ is of bounded variation (Arzela) ([3], [7], p. 114),

(iii) $f(x, y)$ satisfies any one of the many conditions sufficient for the convergence of the double Fourier series given by L. Tonelli ([17], pp. 450-73). In all these conditions he demands that $f(x, y)$ be of b. v. T. plus some other restrictions.

(iv) A condition recently introduced by J. J. Gergen [8] is of the same character.

Hence

THEOREM IV. *The double SL series of an integrable function $f(x, y)$ is convergent at all points (x, y) in Q for which any one of conditions* (i) *to* (iv) *is satisfied. If (x, y) is a point of continuity of $f(x, y)$ the sum of the series is $f(x, y)$. If it is a point of ordinary discontinuity the sum is*

$$\tfrac{1}{4}(f(x+0, y+0)+f(x+0, y-0)+f(x-0, y+0)+f(x-0, y-0)).$$

PART IV. Modified Poisson Summability and its Application to the SL Series.

6. Modified Poisson summability method.

1. Let $f(\theta)$ be integrable on $(-\pi, \pi)$. Applying the well-known *Abel-Poisson summation method* to Fourier series, we multiply its n-th term by r^n, getting the series

$$\tfrac{1}{2}b_0 + \sum_{n=1}^{\infty} (b_n \cos n\theta + b'_n \sin n\theta) r^n = f(\theta, r), \tag{1}$$

which is absolutely and uniformly convergent for $(-\pi \leqq \theta \leqq \pi)$ if $(0 \leqq r < 1)$. Then if $f(\theta, r)$ has a limit as $r \to 1-0$, the Fourier series is said to be summable by this method. Fatou has proved that for any integrable $f(\theta)$

$$\lim_{r \to 1-0} f(\theta, r) = f(\theta) \text{ a. e. } [6]. \tag{2}$$

The analogous theorem is not valid for two dimensions. If it is assumed that $f(\theta, \phi) \log^+ | f(\theta, \phi)|$ is integrable, summability a. e. does follow [14]. It has been shown that the method works also for more general orthonormal systems $\{v_n(\theta)\}$, $(n = 0, 1, 2, \cdots)$, if $f(\theta)$ belongs to L^2 and if suitable restrictions are imposed on the $v_n(\theta)$ ([15], pp. 186-94). (The SL system satisfies these restrictions.)

To get more general results the Poisson method may be modified as follows. Let $v_n(\theta, r)$, $(n = 0, 1, 2, \cdots)$, be the function harmonic in the interior of the unit circle and reducing to $v_n(\theta)$ on the circumference, that is,

$$v_n(\theta, r) = \int_{-\pi}^{\pi} P(\theta, r, s) v_n(s)\, ds, \qquad (n = 0, 1, 2, \cdots), \tag{3}$$

where

$$P(\theta, r, s) = \frac{1}{2\pi} \frac{1 - r^2}{1 - 2r \cos(\theta - s) + r^2} \tag{4}$$

and

$$\lim_{r \to 1-0} v_n(\theta, r) = v_n(\theta) \text{ a. e. (Fatou)}. \tag{5}$$

(For $v_n(s) = \cos ns$, $v_n(\theta, r) = \cos n\theta\, r^n$ and similarly for the sine.) We then form the series

$$\sum_{n=0}^{\infty} c_n v_n(\theta, r), \qquad c_n = \int_{-\pi}^{\pi} v_n(s) f(s)\, ds \qquad (n = 0, 1, 2, \cdots) \tag{6}$$

(the *modified Poisson* ($(m.P.)$) *series of* $f(\theta)$ formed with respect to $\{v_n(\theta)\}$) and consider the limit as $r \to 1 - 0$. *The orthogonal series* $\sum_{n=0}^{\infty} c_n v_n(\theta)$ *of an integrable function* $f(\theta)$ *will be called summable by the* $(m.P.)$ *method if*

$$\lim_{r \to 1-0} \sum_{n=0}^{\infty} c_n v_n(\theta, r) \tag{7}$$

exists. The method reduces to the Abel-Poisson for ordinary Fourier series.

Similarly we may define *summability of the double series* $\sum_{m=0,\, n=0}^{\infty} c_{mn} v_m(\theta) v_n(\phi)$. In this case we consider the limit

$$\lim_{\substack{r \to 1-0 \\ \rho \to 1-0}} \sum_{m=0,\, n=0}^{\infty} c_{mn} v_m(\theta, r) v_n(\phi, \rho), \tag{8}$$

where $v_m(\theta, r)$ and $v_n(\phi, \rho)$ are defined by (3). However it must be remarked that our method could be generalized in several ways to the case of two variables (see [4]). Carrying through the generalization for the double SL system by introducing double harmonic functions on a bicylinder we can prove *equisummability (m. P.) of the double SL and FCC series for an integrable function without any further restriction* (7.2). (For the one variable case the proof is exceedingly simple.) In 7.3 we generalize some of our *results* to *other orthonormal systems.*

2. The modified Poisson method for orthogonal series of functions of L^2 [5].

THEOREM 3. $\sum_{n=0}^{\infty} c_n v_n(\theta, r)$ *converges absolutely and uniformly for all* θ *in* $(-\pi, \pi)$ *and* r *in any closed interval in the interior of* $(0, 1)$.[22]

Proof. By the Schwarz inequality

$$\sum_{n=0}^{\infty} | c_n v_n(\theta, r) | \leqq (\sum_{n=0}^{\infty} c_n^2)^{\frac{1}{2}} (\sum_{n=0}^{\infty} v_n^2(\theta, r))^{\frac{1}{2}}. \tag{9}$$

Then by (3) and Bessel's inequality applied to each factor on the right of (9)

$$\begin{aligned} \sum_{n=0}^{\infty} | c_n v_n(\theta, r) | &\leqq (\int_{-\pi}^{\pi} f^2(s)\, ds)^{\frac{1}{2}} (\int_{-\pi}^{\pi} P^2(\theta, r, s)\, ds)^{\frac{1}{2}} \\ &\leqq \frac{1}{2\pi} \frac{1+r}{1-r} (2\pi)^{\frac{1}{2}} (\int_{-\pi}^{\pi} f^2(s)\, ds)^{\frac{1}{2}}. \end{aligned} \tag{10}$$

THEOREM 4. *Let* $\{v_n(\theta, r)\}$, $(n = 0, 1, 2, \cdots)$ *be an orthonormal system complete with respect to functions of class* L^2. *Then the orthogonal series of* $f(\theta)$ *with respect to this system is equisummable* (m. P.) *with its Fourier series.*

The proof of this theorem, namely, that

$$\lim_{r \to 1-0} | \sum_{n=0}^{\infty} c_n v_n(\theta, r) - f(\theta, r) | = 0, \qquad (-\pi \leqq \theta \leqq \pi), \tag{11}$$

is given in [5] by showing that the (m. P.) series $\sum_{n=0}^{\infty} c_n v_n(\theta, r)$ is equiconvergent with the Poisson integral $f(\theta, r)$ for $(0 \leqq r < 1)$.

7. Application to SL series of functions of class L.

1. Equisummability of the simple SL and FC series. The following theorem is easily proved:

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[22] This proof of Theorem 3 seems to be more direct for real functions than Bergman's proof.

THEOREM V'. *The (m. P.) series* $\sum_{n=0}^{\infty} a_n\phi_n(\theta, r)$ *of any integrable function* $f(\theta)$ *is uniformly equiconvergent on* $(0, \pi)$ *with its Poisson integral* $(0 \leqq r < 1)$. *Uniform equisummability (m. P.) on* $(0, \pi)$ *of the SL series of* f *with its FC series follows.*

Proof. Since the SL functions, $\{\phi_n(\theta)\}$, $(n = 0, 1, 2, \cdots)$, are defined on $(0, \pi)$ we consider

$$\phi_n(\theta, r) = \int_0^{\pi} \bar{P}(\theta, r, s)\phi_n(s)\,ds, \tag{1}$$

where

$$\bar{P}(\theta, r, s) = \frac{1}{2\pi}\left\{\frac{1 - r^2}{1 - 2r\cos(\theta - s) + r^2} + \frac{1 - r^2}{1 - 2r\cos(\theta + s) + r^2}\right\}.^{23} \tag{2}$$

If we suppose that $f(\theta)$ is even, then

$$f(\theta, r) = \int_0^{\pi} \bar{P}(\theta, r, s)f(s)\,ds \qquad \text{(cf. 6 (1))} \tag{3}$$

and, in particular,

$$\cos n\theta\, r^n = \int_0^{\pi} \bar{P}(\theta, r, s)\cos ns\,ds, \qquad 1 = \int_0^{\pi} \bar{P}(\theta, r, s)\,ds. \tag{4}$$

Call

$$\sigma_n(\theta, r) = \sum_{\nu=0}^{n} a_n\phi_\nu(\theta, r), \tag{5}$$

and

$$\sigma'_n(\theta, r) = \tfrac{1}{2}b_0 + \sum_{\nu=1}^{n} b_\nu \cos \nu\theta\, r^\nu. \tag{6}$$

We prove that

$$\lim_{n\to\infty} |\sigma_n(\theta, r) - \sigma'_n(\theta, r)| = 0 \text{ for all } \theta \text{ in } (0, \pi) \text{ and } (0 \leqq r < 1). \tag{7}$$

Using $\Phi_n(s, u)$ defined in 1 (10), we get by direct computation

$$\begin{aligned} |\sigma_n(\theta, r) - \sigma'_n(\theta, r)| &= \left|\int_0^{\pi}\int_x^{\pi} \Phi_n(s, u)\bar{P}(\theta, r, s)f(u)\,du\,ds\right| \\ &= \left|\int_0^{\pi} \bar{P}(\theta, r, s)\left(\int_0^{\pi} \Phi_n(s, u)f(u)\,du\right)ds\right| \text{ (by Fubini)} \\ &\leqq \int_0^{\pi} |\bar{P}(\theta, r, s)|\left|\int_0^{\pi} \Phi_n(s, u)f(u)\,du\right| ds \\ &\leqq \frac{1}{\pi}\frac{1 + r}{1 - r}\int_0^{\pi} \left|\int_0^{\pi} \Phi_n(s, u)f(u)\,du\right| ds. \end{aligned} \tag{8}$$

[23] This corresponds exactly to what we have done for the cosine kernel in 1.

By Haar's theorem this last expression is less than $\epsilon(1+r)/(1-r)$, for $n > N_\epsilon$, ϵ an arbitrary positive number, which proves the theorem.

2. Equisummability of the double SL and FCC series. We now prove the corresponding theorem for the double SL series of an integrable function $f(\theta, \phi)$.

THEOREM V. *The (m. P.) series* $\sum_{m,n=0}^{\infty} a_{mn}\phi_m(\theta, r)\phi_n(\phi, \rho)$ *of any integrable function* $f(\theta, \phi)$ *is equiconvergent with its double Poisson integral over* Q $(0 \leqq r < 1,\ 0 \leqq \rho < 1)$. *Equisummability (m. P.) over* Q *of the double SL series of* f *with its FCC series follows.*

Proof. Corresponding to the notation used in (5) and (6), we let

$$\sigma_{mn}(\theta, \phi; r, \rho) = \sum_{\nu,\mu=0}^{m,n} a_{\nu\mu}\phi_\nu(\theta, r)\phi_\mu(\phi, \rho), \tag{9}$$

and

$$\sigma'_{mn}(\theta, \phi; r, \rho) = \sum_{\nu,\mu=0}^{m,n} \lambda_{\nu\mu} b_{\nu\mu} \cos \nu\theta \cos \mu\phi\, r^\nu \rho^\mu, \tag{10}$$

where $a_{\nu\mu}$, $\lambda_{\nu\mu}$ and $b_{\nu\mu}$ are defined as in **1**, and we prove that

$$\lim_{m\to\infty,\ n\to\infty} |\sigma_{mn}(\theta, \phi; r, \rho) - \sigma'_{mn}(\theta, \phi; r, \rho)| = 0, \tag{11}$$

for all (θ, ϕ) in Q and $(0 \leqq r < 1,\ 0 \leqq \rho < 1)$. Now

$$\sigma_{mn}(\theta, \phi; r, \rho) = \sum_{\nu,\mu=0}^{m,n} a_{\nu\mu} \int_0^\pi \int_0^\pi \bar{P}(\theta, r, s) P(\phi, \rho, t)\phi_\nu(s)\phi_\mu(t)\, ds\, dt \tag{12}$$

$$= \int_0^\pi \int_0^\pi \int_0^\pi \int_0^\pi K_m(s, u) K_n(t, v) \bar{P}(\theta, r, s) \bar{P}(\phi, \rho, t) f(u, v)\, du\, dv\, ds\, dt$$

and similarly

$$\sigma'_{mn}(\theta, \phi; r, \rho) = \sum_{\nu,\mu=0}^{m,n} \lambda_{\nu\mu} b_{\nu\mu} \int_0^\pi \int_0^\pi \bar{P}(\theta, r, s) \bar{P}(\phi, \rho, t) \cos \nu s \cos \mu t\, ds\, dt \tag{13}$$

$$= \int_0^\pi \cdots \int_0^\pi C_m(s, u) C_n(t, v) \bar{P}(\theta, r, s) \bar{P}(\phi, \rho, t) f(u, v)\, du\, dv\, ds\, dt.$$

Let us set (in analogy to **3.** 1)

$$\Lambda^{(1)}_{mn}(\theta, \phi; r, \rho) = \int_0^\pi \cdots \int_0^\pi \Phi_m(s, u) C_n(t, v) \bar{P}(\theta, r, s) \bar{P}(\phi, \rho, t) f(u, v)\, du\, dv\, ds\, dt, \tag{14}$$

$$\Lambda^{(2)}_{mn}(\theta, \phi; r, \rho) = \int_0^\pi \cdots \int_0^\pi C_m(s, u) \Phi_n(t, v) \bar{P}(\theta, r, s) \bar{P}(\phi, \rho, t) f(u, v)\, du\, dv\, ds\, dt, \tag{15}$$

$$\Lambda^{(3)}_{mn}(\theta, \phi; r, \rho) = \int_0^\pi \cdots \int_0^\pi \Phi_m(s, u) \Phi_n(t, v) \bar{P}(\theta, r, s) \bar{P}(\phi, \rho, t) f(u, v)\, du\, dv\, ds\, dt. \tag{16}$$

THEOREM V'. *The* $(m.P.)$ *series* $\sum_{n=0}^{\infty} a_n\phi_n(\theta,r)$ *of any integrable function* $f(\theta)$ *is uniformly equiconvergent on* $(0,\pi)$ *with its Poisson integral* $(0 \leqq r < 1)$. *Uniform equisummability* $(m.P.)$ *on* $(0,\pi)$ *of the SL series of* f *with its FC series follows.*

Proof. Since the SL functions, $\{\phi_n(\theta)\}$, $(n = 0, 1, 2, \cdots)$, are defined on $(0,\pi)$ we consider

$$\phi_n(\theta,r) = \int_0^\pi \bar{P}(\theta,r,s)\phi_n(s)\,ds, \tag{1}$$

where

$$\bar{P}(\theta,r,s) = \frac{1}{2\pi}\left\{\frac{1-r^2}{1-2r\cos(\theta-s)+r^2} + \frac{1-r^2}{1-2r\cos(\theta+s)+r^2}\right\}^{23}. \tag{2}$$

If we suppose that $f(\theta)$ is even, then

$$f(\theta,r) = \int_0^\pi \bar{P}(\theta,r,s)f(s)\,ds \qquad \text{(cf. 6 (1))} \tag{3}$$

and, in particular,

$$\cos n\theta\, r^n = \int_0^\pi \bar{P}(\theta,r,s)\cos ns\,ds, \qquad 1 = \int_0^\pi \bar{P}(\theta,r,s)\,ds. \tag{4}$$

Call

$$\sigma_n(\theta,r) = \sum_{\nu=0}^{n} a_n\phi_\nu(\theta,r), \tag{5}$$

and

$$\sigma'_n(\theta,r) = \tfrac{1}{2}b_0 + \sum_{\nu=1}^{n} b_\nu \cos\nu\theta\, r^\nu. \tag{6}$$

We prove that

$$\lim_{n\to\infty} |\,\sigma_n(\theta,r) - \sigma'_n(\theta,r)\,| = 0 \text{ for all } \theta \text{ in } (0,\pi) \text{ and } (0 \leqq r < 1). \tag{7}$$

Using $\Phi_n(s,u)$ defined in **1** (10), we get by direct computation

$$\begin{aligned}
|\,\sigma_n(\theta,r) - \sigma'_n(\theta,r)\,| &= \left|\int_0^\pi\int_x^\pi \Phi_n(s,u)\bar{P}(\theta,r,s)f(u)\,du\,ds\right| \\
&= \left|\int_0^\pi \bar{P}(\theta,r,s)\left(\int_0^\pi \Phi_n(s,u)f(u)\,du\right)ds\right| \text{ (by Fubini)} \\
&\leqq \int_0^\pi |\,\bar{P}(\theta,r,s)|\left|\int_0^\pi \Phi_n(s,u)f(u)\,du\right|ds \\
&\leqq \frac{1}{\pi}\frac{1+r}{1-r}\int_0^\pi \left|\int_0^\pi \Phi_n(s,u)f(u)\,du\right|ds.
\end{aligned} \tag{8}$$

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[23] This corresponds exactly to what we have done for the cosine kernel in **1**.

By Haar's theorem this last expression is less than $\epsilon(1+r)/(1-r)$, for $n > N_\epsilon$, ϵ an arbitrary positive number, which proves the theorem.

2. Equisummability of the double SL and FCC series. We now prove the corresponding theorem for the double SL series of an integrable function $f(\theta, \phi)$.

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Proof. Corresponding to the notation used in (5) and (6), we let

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and

$$\sigma'_{mn}(\theta, \phi; r, \rho) = \sum_{\nu,\mu=0}^{m,n} \lambda_{\nu\mu} b_{\nu\mu} \cos \nu\theta \cos \mu\phi\, r^\nu \rho^\mu, \tag{10}$$

where $a_{\nu\mu}$, $\lambda_{\nu\mu}$ and $b_{\nu\mu}$ are defined as in **1**, and we prove that

$$\lim_{m\to\infty,\, n\to\infty} |\sigma_{mn}(\theta, \phi; r, \rho) - \sigma'_{mn}(\theta, \phi; r, \rho)| = 0, \tag{11}$$

for all (θ, ϕ) in Q and $(0 \leqq r < 1,\ 0 \leqq \rho < 1)$. Now

$$\begin{aligned}\sigma_{mn}(\theta, \phi; r, \rho) &= \sum_{\nu,\mu=0}^{m,n} a_{\nu\mu} \int_0^\pi \int_0^\pi \bar{P}(\theta, r, s) P(\phi, \rho, t)\phi_\nu(s)\phi_\mu(t)\, ds dt \\ &= \int_0^\pi \int_0^\pi \int_0^\pi \int_0^\pi K_m(s, u) K_n(t, v) \bar{P}(\theta, r, s) \bar{P}(\phi, \rho, t) f(u, v)\, du dv ds dt\end{aligned} \tag{12}$$

and similarly

$$\begin{aligned}\sigma'_{mn}(\theta, \phi; r, \rho) &= \sum_{\nu,\mu=0}^{m,n} \lambda_{\nu\mu} b_{\nu\mu} \int_0^\pi \int_0^\pi \bar{P}(\theta, r, s) \bar{P}(\phi, \rho, t) \cos \nu s \cos \mu t\, ds dt \\ &= \int_0^\pi \cdots \int_0^\pi C_m(s, u) C_n(t, v) \bar{P}(\theta, r, s) \bar{P}(\phi, \rho, t) f(u, v)\, du dv ds dt.\end{aligned} \tag{13}$$

Let us set (in analogy to **3.** 1)

$$\Lambda^{(1)}_{mn}(\theta, \phi; r, \rho) = \int_0^\pi \cdots \int_0^\pi \Phi_m(s, u) C_n(t, v) \bar{P}(\theta, r, s) \bar{P}(\phi, \rho, t) f(u, v)\, du dv ds dt, \tag{14}$$

$$\Lambda^{(2)}_{mn}(\theta, \phi; r, \rho) = \int_0^\pi \cdots \int_0^\pi C_m(s, u) \Phi_n(t, v) \bar{P}(\theta, r, s) \bar{P}(\phi, \rho, t) f(u, v)\, du dv ds dt, \tag{15}$$

$$\Lambda^{(3)}_{mn}(\theta, \phi; r, \rho) = \int_0^\pi \cdots \int_0^\pi \Phi_m(s, u) \Phi_n(t, v) \bar{P}(\theta, r, s) \bar{P}(\phi, \rho, t) f(u, v)\, du dv ds dt. \tag{16}$$

We wish to prove that the three integrals $|\Lambda^{(1)}_{mn}|$, $|\Lambda^{(2)}_{mn}|$ and $|\Lambda^{(3)}_{mn}| \to 0$ as $m, n \to \infty$ for all (θ, ϕ) in Q and $(0 \leqq r < 1,\ 0 \leqq \rho < 1)$.

To consider $\Lambda^{(1)}_{mn}(\theta, \phi; r, \rho)$ proceed as in inequality (8) and then

$$(17)\quad |\Lambda^{(1)}_{mn}(\theta, \phi; r, \rho)|$$

$$\leqq \int_0^\pi \int_0^\pi |\bar{P}(\theta, r, s)|\,|P(\phi, \rho, t)|\,\Big|\int_0^\pi \int_0^\pi \Phi_m(s, u)\Phi_n(t, v) f(u, v)\,du dv\Big|$$

$$\leqq \frac{1}{\pi}\,\frac{1+r}{1-r}\,\frac{1+\rho}{1-\rho}\int_0^\pi \int_0^\pi \Big|\int_0^\pi \int_0^\pi \Phi_m(s, u)\Phi_n(t, v) f(u, v)\,du dv\Big|\,ds dt.$$

But we have proved in **5.** 2 that the inner integral on the right side converges to 0 uniformly with respect to (s, t) in Q as $m, n \to \infty$, so that

$$(18)\qquad |\Lambda^{(1)}_{mn}(\theta, \phi; r, \rho)| < \frac{1+r}{1-r}\,\frac{1+\rho}{1-\rho}\,\epsilon \text{ for } \frac{m}{n} > N_\epsilon,$$

ϵ an arbitrary positive number.

Considering $\Lambda^{(2)}_{mn}(\theta, \phi; r, \rho)$ we have that

$$(19)\quad |\Lambda^{(2)}_{mn}(\theta, \phi; r, \rho)|$$

$$= \Big|\int_0^\pi \int_0^\pi f(u, v)\Big(\int_0^\pi C_m(s, u)\bar{P}(\theta, r, s)\,ds\Big)\Big(\int_0^\pi \Phi_n(t, v)\bar{P}(\phi, \rho, t)\,dt\Big)du dv\Big|$$

$$\leqq \int_0^\pi \int_0^\pi |f(u, v)|\,\Big|\int_0^\pi C_m(s, u)\bar{P}(\theta, r, s)\,ds\Big|\,\Big|\int_0^\pi \Phi_n(t, v)\bar{P}(\phi, \rho, t)\,dt\Big|$$

Now by formula (4)

$$(20)\qquad \Big|\int_0^\pi C_m(s, u)\bar{P}(\theta, r, s)\,ds\Big| = \Big|\tfrac{1}{2} + \sum_{\nu=1}^{n} \cos \nu u \cos \nu\theta\, r^\nu\Big|$$

$$\leqq \frac{1}{1-r}.$$

Secondly by Haar's theorem, for arbitrary positive ϵ,

$$(21)\qquad \Big|\int_0^\pi \Phi_n(t, v)\bar{P}(\phi, \rho, t)\,dt\Big| < \epsilon \text{ for } n > N = N_\epsilon(\phi, \rho),$$

so that

$$(22)\quad |\Lambda^{(2)}_{mn}(\theta, \phi; r, \rho)| \leqq \frac{1}{1-r}\,\epsilon \int_0^\pi \int_0^\pi |f(u, v)|\,du dv, \text{ for } n > N \text{ and all } m.$$

Hence

$$(23)\quad \lim_{m\to\infty,\ n\to\infty} |\Lambda^{(2)}_{mn}(\theta, \phi; r, \rho)| = 0, \text{ for all } (\theta, \phi) \text{ and } (0 \leqq r < 1,\ 0 \leqq \rho < 1).$$

Similarly for $|\Lambda^{(3)}_{mn}(\theta, \phi; r, \rho)|$ and the theorem is proved.

3. Remark on the modified Poisson method for more general orthonormal systems. The applicability of this method is not limited to the SL

orthonormal system. We have seen in **6.2** that the (m. P) series of any function of class L^2 with respect to a complete orthonormal system coincides with the Poisson integral of f. If we assume that the function $f(\theta)$ is of class L, we can prove the following theorem.

THEOREM VI. *Let* $\{v_n(\theta)\}$, $(n = 0, 1, 2, \cdots)$, *be a complete orthonormal system and let* $L_n(\theta, s) = \sum_{\nu=0}^{n} v_\nu(\theta) v_\nu(s)$. *If the condition*

$$(24) \qquad \left| \int_{-\pi}^{h} L_n(\theta, s)\, ds \right| \leqq M, \qquad (-\pi \leqq h \leqq \pi),^{24}$$

(M independent of h, n and θ*), is satisfied, then the (m. P) series of an integrable function* $f(\theta)$ *is equiconvergent with its Poisson integral and the development of* $f(\theta)$ *is equisummable (m. P.) with its Fourier series.*

The proof of this theorem will be given on another occasion. This theorem can be generalized in various ways to more variables.

BRYN MAWR COLLEGE.

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LINEAR TOPOLOGICAL SPACES.*[1]

By D. G. BOURGIN.

1. Introduction.[2] A (real) linear topological space, l. t. s., is a linear Hausdorff space in which the operations of addition and multiplication by a real number are continuous. Axioms sufficient to ensure that a space is a l. t. s. have been given by Kolmogoroff [1] and von Neumann [2] and the relation between these axioms has often been discussed [3(2), 4]. For convenience in reference we state an axiom system which, with the exception of axiom 1, [3], is that given by von Neumann.

We understand by $\mathfrak{N}$ a neighborhood basis $\{N\}$ of the origin, θ, in L. For this neighborhood basis and for *all bases* introduced in this work (including those for the conjugate space as well) the basis at a point not the origin is obtained by a translation. Thus the basis at x_0 is $\mathfrak{N} + x_0$.

I. $\cap \{N \mid N \in \mathfrak{N}\} = \theta$.

II. If $N_1, N_2 \in \mathfrak{N}$ there is an $N_3 \in \mathfrak{N}$ such that $N_3 \subset N_1 \cap N_2$.

III. For any $N \in \mathfrak{N}$ there is an $N_1 \in \mathfrak{N}$ such that $\alpha N_1 \subset N$ for all $|\alpha| \leq 1$.

IV. For any $N \in \mathfrak{N}$ there is an $N_1 \in \mathfrak{N}$ such that $N_1 + N_1 \subset N$.

V. For arbitrary $x \in L$ and $N \in \mathfrak{N}$ there is a real number α such that $x \in \alpha N$.

The symbol L with null element θ will henceforth denote a l. t. s., i. e., a space satisfying these axioms. If L is *locally convex*, then

VI. $N + N \subset 2N$.

A set $B \subset L$ is bounded if for each $N \in \mathfrak{N}$ there is a real number α such that $B \subset \alpha N$ and L is *locally bounded* if it contains a bounded open set. In view of Axiom III, α may be taken non-negative in Axiom V and in the definition of boundedness.

We denote the closed hull of A by $\bar{A}$, the convex hull by $\hat{A}$, the set of inner points in A by $\underline{A}$ and the complement of A by $\tilde{A}$. The set of all linear functionals on L is designated by L^* and its hull and general elements are denoted by θ^* and f or F respectively. We write $\bar{L}$ and $\underline{L}$ for the lim sup and

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[2] Numbers in brackets refer to the bibliography.

lim inf respectively and ϕ for the empty set. The Euclidean n dimensional space is denoted by R_n.

A brief partial survey of some of the contents follows. Some general results for convex sets are developed first and these in part involve regular L convexity and weak closure properties. For locally convex spaces, regular L convexity is no stronger than closure and convexity. Although usually L is no longer an l. t. s. in its weak topology, we can associate a new space of equivalence classes of L which is a locally convex l. t. s. Most of the conclusions involve specialization of L. Thus if L is locally convex there is a valid analogue of Mazur's Theorem [5] on the equivalence of strong and weak closure of convex sets in Banach spaces, and a result of the type of Helly's Theorem obtains if we require besides that L^* be metrizable. If L^* is normable then L is the union of a denumerable number of multiples of some bounded set. A necessary and sufficient condition for the metrizability of L^* is given in terms of the existence of a certain basis for the bounded sets in L. The sufficiency condition is valid for a general l. t. s. If L is locally bounded (locally convex) its completion is locally bounded (locally convex) and the conjugate spaces of L and its completion are equivalent in the Banach sense. The results of Hyers [3a] are extended and sharpened significantly. Thus it is shown that the quasi norm satisfies conditions similar to those for the spaces $L_p, 0 < p < 1$. The exact conditions depend on the choice of the fundamental bounded open set and in general the quasi norm need not be continuous. It is shown, nevertheless, that there always exists a topologically equivalent continuous quasi norm. The weak topology of the conjugate space of a locally bounded l. t. s. yields many results of the sort familiar for vector normed spaces. For instance regular L^* convexity is equivalent to weak closure and, the unit sphere is bicompact in the weak L^* topology, etc. However, the exact relation between transfinite closure and weak closure is not yet settled in this paper.

2. The general linear topological space. In the main, the results in this section require no specialization of L and are, besides, of use in later sections. It is convenient to introduce the neighborhood systems

$$\mathfrak{U} = \{U\}, \qquad U = \{\lambda x \mid 0 \leq \lambda \leq 1, x \in N\},$$

and $U^s = U_\omega - U$. In referring to neighborhoods in L or L^* we shall omit the basis when the meaning is plain. Thus we write U for $U \in \mathfrak{U}$ and N for $N \in \mathfrak{N}$ etc. We assume throughout the paper that L contains more than one element. It is easy to see that equivalent bases $\mathfrak{N}'$ may be chosen in which the

neighborhoods are all open (closed) sets. (In the first case the sets U or U^s are, accordingly, open).

LEMMA 1. *The neighborhood systems* $\mathfrak{U}$, $\mathfrak{U}^s$ *and* $\mathfrak{N}$ *are equivalent neighborhood bases for* L.

It is trivial that an arbitrary U contains an N. On the other hand for arbitrary N and some N_1 we have $\alpha N_1 \subset N$ for $|\alpha| \leq 1$. Hence

$$U_1 = \{\lambda x \mid 0 \leq \lambda \leq 1, x \in N_1\} \subset \{\lambda x \mid -1 \leq \lambda \leq 1, x \in N_1\} \subset N.$$

That $U(U^s)$ satisfies the axioms for an l. t. s. may now be shown directly. If $\mathfrak{N}$ satisfies VI this need not be true of all equivalent bases. A not altogether trivial example is furnished by l_1 which may easily be shown to admit the equivalent metric $(x, \theta) = \log \Pi_1^{\infty}(1 + |x_i|)$. Here $N' = \{x \mid (x, \theta) < \epsilon\}$ is out of accord with VI. A more natural example is the locally convex space s [8] whose metric spheres are plainly not convex. However the convexifications of the equivalent neighborhoods do satisfy VI of course and are understood in all references to locally convex spaces, below.

LEMMA 2. *The space* L *is completely regular.*

An l. t. s. is evidently a uniform space [6] or struct [7] whence the lemma follows immediately. For the actual applications in this paper merely the weaker, regularity property of L is required, in the main.

Let L_α, $\alpha \in \mathfrak{A}$ be an l. t. s. with neighborhood basis $\mathfrak{N}_\alpha$. The topology of the product space $L^{\mathfrak{A}} = \Pi L_\alpha$ is given by the neighborhoods

$$N = \Pi\{N_i \mid i \in \sigma\} \times \Pi\{L_\beta \mid \beta \in \mathfrak{A} - \sigma\}$$

where $N_i \in \mathfrak{N}_i$ and σ is a finite subset of $\mathfrak{A}$. It is well known that $L^{\mathfrak{A}}$ is an l. t. s. Obviously if each L_α is locally convex, $L^{\mathfrak{A}}$ is locally convex.

LEMMA 3. *If* L_α *is locally bounded then a necessary and sufficient condition for* $L^{\mathfrak{A}}$ *to be locally bounded is that* $\mathfrak{A}$ *is finite.*

Evidently local boundedness implies that for some N, $\{N/m, m = 1, 2 \cdots\}$ constitutes a neighborhood basis. Since $\mathfrak{A} - \sigma$ is non-empty, for non-finite $\mathfrak{A}$, this necessary condition cannot be met. An immediate consequence is that if the spaces L_α are Banach spaces and $\mathfrak{A}$ is non-finite then $L^{\mathfrak{A}}$ is not a Banach space. If $\mathfrak{A}$ is finite then if G_α is open ΠG_α is open in $L^{\mathfrak{A}}$ and if besides G_α contains θ and is bounded then the neighborhood basis in $L^{\mathfrak{A}}$ is given by $\{(1/n)\Pi G_\alpha\}$.

We shall say that U encircles θ if every half ray with θ as an end point has an interval in common with U. That the encirclement properly imposes no restrictions on the l. t. s. is evident from the next lemma.

LEMMA 4. *Every N encircles θ.*

From the definition of U it is patent that if $x_0 \epsilon U$ then the interval from θ to x_0 is contained in U. Since N contains some U, if the lemma were false there would be an $\bar{x}$ such that $\lambda \bar{x} \bar{\epsilon} U$ for all $\lambda > 0$ in contradiction with V.

LEMMA 5. *If A is closed and $x_0 \epsilon L$ is disjunct from A then, for some U' and U'', $(A + \bar{U}') \cdot (x_1 + U'') = \phi$.*

Since A is closed we can find U_0^s such that $(x_0 + U_0^s) \cdot A = \phi$. In view of I and IV, $U_0^s \supset U_1^s + U_2 - U_2$ for some U_1^s and U_2. We assert $(A + U_2) \cdot (x_0 + U_1^s) = \phi$. Otherwise for some $x_1 \epsilon A$, $y \epsilon U_1^s$ we should have $x_0 + y_1 \epsilon x_1 + U_2$. This implies successively $x_1 - (x_0 + y_1) \epsilon - U_2$ and $x_1 \epsilon x_0 + y_1 - U_2 \subset x_0 + U_1^s + U_2 - U_2$ in contradiction with the fact that A is disjunct from $x_0 + U_0^s$. In view of Lemma 2 we can find U_3 such that $\bar{U}_3 \subset U$ and hence $(A + \bar{U}_3) \cdot (x_0 + U_1) = \phi$.

LEMMA 6. *If K_1 and K_2 are convex then $K_1 \pm K_2$ is convex.*

LEMMA 7. *If K is convex then $\bar{K}$ and $\underline{K}$ are convex.*

By a translation we can arrange to have $\theta \epsilon K$. Assume the first assertion false. Suppose, then, that $x_1, x_2 \epsilon \bar{K}$ and yet $z = \lambda x_1 + (1 - \lambda) x_2 \epsilon \bar{K}$ for some λ, $0 < \lambda < 1$. Then for some U, $(z + U) \cdot K = \phi$. By III and IV we can find U' to satisfy $\lambda U' + (1 - \lambda) U' \subset U$. Let $y_i \epsilon (x_i + U') \cdot K \neq \phi$. $i = 1, 2$. Evidently λy_1 and $(1 - \lambda) y_2 \epsilon K$. Hence we arrive at the contradiction $\phi \neq [\lambda (x_1 + U') + (1 - \lambda)(x_2 + U')] \cdot K \subset [\lambda x_1 + (1 - \lambda) x_2 + U] \cdot K = \phi$.

Let $x_1, x_2 \epsilon K$. With no loss of generality we may assume $x_1 = \theta$. Let U_1 and $x_2 + U_2$ be contained in K. Suppose that for some λ. $0 < \lambda < 1$, $\lambda x_2 \bar{\epsilon} K$. Let $U_3 \subset U_1 \cdot U_2$. Choose y arbitrarily in $\lambda x_2 + U_3$. Thus $y = \lambda x_2 + z$, $z \epsilon U_3$. Clearly z and $z + x_2$ are both in K. Moreover $(1 - \lambda) z + \lambda (z + x_2) = y$ is in K by the convexity of K. Hence $\lambda x_2 + U_3 \subset K$ or $\lambda x_2 \epsilon K$ in contradiction with an earlier assumption.

LEMMA 8. *If $K \neq L$ is convex and $\theta \epsilon \underline{K}$ there is a triangular pseudo norm*[3] $\| x \|_K$ *satisfying* (1) $0 = \| \theta \|_K \leq \| x \|_K < \infty$, (2) $\| x \|_K \leq 1$ *for*

[3] If $x_0 \epsilon K$ we define $\| x \|_{K, x_0}$ by $\inf \{ h > 0 \mid x \epsilon h (K - x_0) \}$ etc.

$x \in K$, (3) $\| t x \|_K = t \| x \|_K, t \geq 0$, (4) $\| x_1 + x_2 \|_K \leq \| x_1 \|_K + \| x_2 \|_K$, (5) *The frontier points are those for which*

$$\| x \|_K = 1, \quad (6)\ \bar{K} = \{x \mid \| x \|_K \leq 1\}, \quad \underline{K} = \{x \mid \| x \|_K \leq 1\}.$$

We define $\| x \|_K$ as $\inf \{h > 0 \mid x \in hK\}$. Let $U \subset K$, then for any $x \in L$ we have $x \in \alpha U$ for some $\alpha > 0$. Clearly $\infty > \alpha \geq \| x \|_K \geq \| \theta \|_K = 0$. Since $\| x \|_K$ is merely the Minkowski functional determined by K, properties (2), (3) and (4) follow by standard arguments.

We have left (5). In contradistinction to the locally convex case, this is not intuitively so evident when one considers the possibilities inherent in neighborhoods with spines. However the proof is direct. Let $U_1^s \subset K$, then, (for arbitrary $\epsilon > 0$,) in view of III and IV, $U_1^s \supset (2/\epsilon) U_2^s$ for some U_2^s. Then $\| \pm x \|_K < \epsilon$ for $x \in U_2^s$ and so $\| \pm x \|_K$ is continuous at θ. Hence by (4), $\| \pm x \|_K$ is everywhere continuous.[4] If x_1 is a frontier point, every neighborhood of x_1 contains points of K and points of $\bar{K}$. Thus every neighborhood of x_1 contains some points for which $\| x \|_K \leq 1$ and some for which $\| x \|_K \geq 1$. Therefore $\| x_0 \|_K = 1$. On the other hand if $\| x_0 \|_K = 1$ and U is arbitrary, then by Lemma 4, there is a $\delta > 0$ such that $(1 \pm \delta) x_0 \in x_0 + U$. Hence by (3), $(1 + \delta) x_0 \bar{\in} K$, $(1 - \delta) x_0 \in K$ or x_0 is a frontier point. The forms of $\bar{K}$ and $\underline{K}$ are now seen to be those given in (6).

Extending a familiar usage we shall say K is *regularly L convex* if for every $x_0 \bar{\in} K$ there is an $f_0 \in L^*$ such that $f_0(x_0) > \sup_K f_0(x)$.

LEMMA 9. *If K is regularly L convex then K is convex.*

THEOREM 1. *If $K \neq L$ is convex, closed and $\underline{K} \neq \emptyset$ then K is regularly L convex.*

It is no loss of generality to take θ as an inner point. Let $x_0 \bar{\in} K$. The linear space L_1 is defined by $\{x \mid x = t x_0, -\infty < t < \infty\}$. Write $f_0(t x_0) = t \| x_0 \|_K$ for all t. Since $\| t x_0 \|_K \geq f(t x_0)$, the function $\| x \|_K$ is available for $p(x)$ in the Banach statement of the Hahn-Banach Theorem [8; p. 27]. Thus there is a distributive functional F_0 on L which agrees with f_0 on L_1 and $F_0(x) = \| x \|_K$. Now $| F_0(x) | \leq \sup (\| x \|_K, \| -x \|_K)$. In connection with (5) of the previous lemma we demonstrated the continuity of $\| x \|_K$ and $\| -x \|_K$. Thus $F_0(x)$ is continuous at $x = \theta$ and so throughout L. Hence choosing F_0 so that $F_0(x_0) = \| x_0 \|_K$ it follows from (5) that

[4] The argument used to establish such a result for the case of locally convex spaces [2] makes unnecessary use of VI and so could not be cited directly.

$$F_0(x_0) > 1 \geq \sup_K \| x \|_K \geq \sup_K F_0(x).$$

(it is trivial that every non null linear functional on L may be related to a Minkowski functional. Thus let $K = \{x \mid |f(x)| \leq 1\}$ and plainly $\theta \varepsilon K$ since $\{x \mid |f(x)| < 1\}$ is open. Moreover either $f(x_0) = \| x_0 \|$ or $f(-x_0) = \| -x_0 \|$ for the frontier points $\pm x_0$).

LEMMA 10. *If $K \neq L$ is convex, and $\theta \varepsilon \underline{K}$, then $\{x \mid \| x \|_K \leq 1\}$ is regularly L convex.*

3. The weak topology of L. The generic $w(L)$ neighborhood of θ is denoted by $W(\theta; f_1, \cdots, f_n; \epsilon)$ (or merely W) $= \{x \mid |f_i(x)| < \epsilon, i = 1, \cdots, n\}$, where $\{f_i\}$ consists of linearly independent elements of L^* in the sequel. The neighborhoods of x_0 are then given by $x_0 + W$. In general it is not true that the system of weak neighborhoods, $W = \{W\}$, is consistent with the axioms for a l. t. s. For instance in the space S or $L^{\frac{1}{2}}$ it is well known that L^* consists only of the null element θ^*. Hence I is not satisfied.

Let $\Theta = \{x \mid f(x) = 0 \text{ for all } f \varepsilon L^*\}$ and $X = x + \Theta$. The sets Θ, X considered as elements of a new space $\boldsymbol{L}$ are designated by $\boldsymbol{O}$ and $\boldsymbol{X}$ respectively. Evidently if $x \varepsilon W$ then $X = x + \Theta \varepsilon W$ and accordingly W is made up of distinct X sets. Since Θ is closed we define a natural basis at $\boldsymbol{O}$ by

$$\boldsymbol{W}(\boldsymbol{O}; f_1, \cdots, f_n; \epsilon) = \{\boldsymbol{X} \mid X \varepsilon W(\theta; f_1, \cdots, f_n; \epsilon\}.$$

The topology so defined is tacitly understood in subsequent references to $\boldsymbol{L}$. Manifestly then, $\boldsymbol{L}$ is a space of disjunct equivalence classes defined on L.

THEOREM 2. *The space $\boldsymbol{L}$ is a locally convex* l. t. s.

In view of Theorem 3 b kx_1 and $x_1 + x_2$ correspond to $k\boldsymbol{X}_1$ and $\boldsymbol{X}_1 + \boldsymbol{X}_2$ where $x_i \varepsilon X_i, i = 1, 2$ so that $\boldsymbol{L}$ is a linear space. Write $\boldsymbol{X} = Tx$. Obviously T is continuous since $\boldsymbol{W} = T(W)$. Now I and II are satisfied since $\boldsymbol{L}$ is obviously the Hausdorff topological quotient group L/Θ with addition as the group operation. Now

$$\alpha \boldsymbol{W}(\boldsymbol{O}; f_1, \cdots, f_n; \epsilon) = \boldsymbol{W}(\boldsymbol{O}; f_1, \cdots, f_n; |\alpha| \epsilon) \varepsilon \boldsymbol{W}(\boldsymbol{O}; f_1, \cdots, f_n; \epsilon)$$

for $|\alpha| \leq 1$, or III is satisfied. Since $|f(x_1 + x_2)| \leq |f(x_1)| + |f(x_2)|$ it is apparent that IV and VI hold. Finally suppose $x_0 \varepsilon X_0$, and let $c_i = f_i(x_0)$, $i = 1, \cdots, n$. If $a > \sup |c_i|$ then $x_0 \varepsilon (a/\epsilon) W(\theta; f_1, \cdots, f_n; \epsilon)$ and so $\boldsymbol{X}_0 \epsilon (a/\epsilon) \boldsymbol{W}(\boldsymbol{O}; f_1, \cdots, f_n; \epsilon)$ which is the requirement of V.

COROLLARY. *If $\Theta = \theta$, L is locally convex in its $w(L)$ topology.*

This includes the case of a locally convex L.

It is easy to give examples of linear topological spaces for which Θ is neither θ nor L. Since $L^*_p = \theta^*$, $0 < p < 1$ [9] take

$$L = L^p \times R_1 = \{(x, y) \mid x \in L^p, y \in R_1\}.$$

Then $\Theta = \{(x, \theta)\}$.

THEOREM 3. *The set Θ has the following properties:* (a) *Θ is convex and closed.* (b) *$\alpha\Theta = \Theta$ for $\alpha \neq 0$.* (c) *If P is a linear operator on L_1 to L_2 then $P(\Theta_1) \subset \Theta_2$.* (d) *If $L^* \neq \theta^*$ then $\Theta = \phi$.* (e) *For regularly L convex K, either K or $\bar{K}$ contains X.* (f) *If L is convex $\Theta = \theta$.*

The first three assertions are obvious. Suppose (d) false. Let $x_1 + U_1 \subset \Theta$. Then x_1 and so U_1 also, are both in Θ. Hence by (b) and V, $L \subset \Theta$ or $L^* = \theta^*$. If (e) is false then x_1 and x_2 may be found in X with $f(x_1) > f(x_2)$ in contradiction with the definition of X. Assertion (f) is a trivial consequence of Lemma 2, and Theorem 1. The converse of (f) is false. For instance in a sequence space the functionals $f^n = (\delta_1{}^n, \cdots, \delta_i{}^n, \cdots)$, with $\delta_i{}^n$ the Kronecker δ, cannot all be zero for any x. Hence, for instance, the non-locally convex l_p, $0 < p < 1$ has $\Theta = \theta$. (We remark parenthetically that while the metrizability of l_p, $0 < p < 1$, has long been recognized, the trivial observation that a valid metric for this purpose is $\Sigma \mid x_i \mid^p$ seems to have been overlooked. With this metric, it is at once evident that l_p is locally bounded as is well known. Moreover a familiar type of argument establishes that l_p is complete).

THEOREM 4. *The following conditions on $K \subset L$ are equivalent.* (a) *K is regularly L convex.* (b) *K is convex and $w(L)$ closed.*

That (a) implies (b) is trivial. The demonstration of the converse is of conventional type. Thus if $x_0 \bar{\in} K$ let $W(x_0; f_1, \cdots, f_n; \epsilon)$ be disjunct from K. We map L on R_n where the correspondence is given by

$$z = \psi(x) = \{f_1(x), \cdots, f_n(x)\}.$$

Then K_n, the map of K, is convex. Now $z_0 = \psi(x_0)$ is not in $\bar{K}_n$, the closure of K_n in R_n. Indeed if $\| z - z_0 \| < \epsilon$ contains points of K_n then $\Sigma \mid f_i(x) - f_i(x_0) \mid^2 < \epsilon^2$ for some $x \in K$ and accordingly $\mid f_i(x) - f_i(x_0) \mid < \epsilon$ but this is impossible since W and K are disjunct. Some functional $\mathfrak{F}_0 \in R^*_n$

satisfies $\mathfrak{F}_0(z_0) > \sup_{Kn} \mathfrak{F}_0(z)$. Now $\mathfrak{F}_0(z)$ is of the form $\Sigma\lambda_i f_i(x)$ and hence with $F_0 = \Sigma_1^n \lambda_i f_i$ we have

$$\mathfrak{F}_0(z_0) = F_0(x_0) > \sup_{Kn} \mathfrak{F}_0(z) = \sup_K F_0(x).$$

Evidently (a) implies that $T(K)$ is closed and convex.

THEOREM 5. *If $\Theta = \theta$ and K is a closed finite dimensional set then K is regularly L convex.*

Let $\{x_i \mid i = 1, \cdots, N\} \subset L$ be linearly independent. Since $\Theta = \theta$ the following steps are valid. Choose $f_1 \varepsilon L^*$ so that $f_1(x_1) = a_1{}^1 \neq 0$. Write $f_1(x_2) = a_2{}^1$. Then $f_1(a_2{}^1 x_1 - a_1{}^1 x_2) = 0$. For some f_2, $f_2(a_2{}^1 x_1 - a_1{}^1 x_2) \neq 0$. Write $f_2(x_i) = a_i{}^2$, $i = 1, 2$. Hence $\Delta = a_2{}^1 a_1{}^2 - a_1{}^1 a_2{}^2 \neq 0$ and so two linear combinations of f_1 and f_2, designated by F^1 and F^2, satisfy $F^i(x_j) = \delta_j^i$, $i, j = 1, 2$. An induction shows that we can find $\{f^i\}$ to satisfy $f^i(x_j) = \delta_j^i$, $i, j = 1, \cdots, N$, whence it follows incidentally that we can always satisfy $f(x_i) = c_i$, $i = 1, \cdots, N$, for arbitrary choice of $\{c_i\}$. Let the N dimensional manifold E_N be the closed linear extension of K, with basis $\{x_i \mid i = 1, \cdots, N\}$. If $x_0 \bar{\varepsilon} E_N$ then we can find f_0 to satisfy $f_0(x_i) = \delta_0^i$, $i = 0, 1, \cdots, N$. If $x_0 \varepsilon E_N \cap \bar{K}$ then, for some U, $x_0 + U_N \cap K = \emptyset$, $U_N = U \cap E_N$. Now

$$Q = \{x \mid x = \Sigma_1^N \mu_i x_i, \Sigma_1^N \mu_i \mid \leq 1\}$$

is a symmetric convex set in E_N. It is well known that E_N is homeomorphic to R_N with Q the homeomorph of a convex bounded set containing an open set in R_N. Hence $\epsilon Q \subset U_N$ if $\epsilon_0 \geqq \epsilon > 0$. Write $x_0 = \Sigma_1^N \lambda'_i x_i$. If $|\lambda_i - \lambda'_i| < \epsilon_0/N$, $i = 1, \cdots, N$, obviously $x - x_0 = \Sigma(\lambda_i - \lambda'_i) x_i \varepsilon \epsilon Q \subset U_N$. Thus $W(x_0; f^1, \cdots, F^n, \epsilon_0/N) \cap K = \emptyset$ since $f^j(x - x_0) = \lambda_j - \lambda'_j$, and hence in view of Theorem 4 our proof is complete.

The following example shows that $\Theta = \theta$ does not assure that all closed convex sets are regularly L convex. Define K in $l_{1/2}$ by $\{x \mid \| x \| \geq 1, x_i \geq 0\}$. Then $K = \bar{K}$. To see that K is convex consider any segment H with end points x, y in K. We may require $\| x \| = \| y \| = 1$ for if $z \varepsilon H \cap \bar{K}$ then, by continuity, H would contain x', y' of norm 1 with the order x, x', z, y', y. It is easy to verify that $\| \lambda x + 1 - \lambda y \| - 1 = \Sigma'[(A_{ij}{}^2 + \lambda(\lambda - 1) \times (x_i y_j (1 - x_j y_i) + x_j y_i (1 - x_i y_j))^{1/2} - A_{ij}]$, $A_{ij} = \lambda(x_i x_j)^{1/2} + (1 - \lambda) \times (y_i y_j)^{1/2}$. Terms with $i = j$ are omitted and the typical term is clearly non-negative. Hence $H \subset K$. Now $\theta \bar{\varepsilon} K$ yet $f(\theta) = 0 \leq \sup_K f(x)$ for all f. Indeed write $f = (a_1, a_2, \cdots)$ with $a_i \leq 0$ for this is the only case in doubt.

Write $x^N = \Sigma_1{}^N y^i/N^2$, $y^i = (\delta_1{}^i, \delta_2{}^i, \cdots)$. Then $\| x^N \| = 1$ and $x^N \varepsilon K$. Now $l_{1/2}{}^*$ is the space in [8, 9]. Thus $-\infty < a \leq \inf a_i \leq 0$ and $0 \geq f(x^N) = \Sigma_i a_i x_i{}^N \geq a/N \to 0$. Hence K is not regularly L convex. Incidentally $\underline{K} = \phi$.

THEOREM 6. *If $A \subset L$ is $w(L)$ closed then A is closed.*

Suppose $x_0 \bar{\varepsilon} A$ and let $W(x_0; f_1, \cdots, f_n; \epsilon) \cap A = \emptyset$. Since f_i is continuous, $|f_i(x - x_0)| < \epsilon$ for all $x \varepsilon x_0 + U_i$. Thus $x_0 + \cap_1{}^n U_i \subset W$.

4. Locally convex spaces. The hypothesis of local convexity is *understood* in the statements of the results in this section.

LEMMA 11. *The completion of L is a locally convex* l. t. s.

We assume the material in Tukey's Chapter VI [7] but translate coverings into neighborhoods by his Lemma 3. 3. Thus the equivalence classes of Cauchy phalanxes in L are put in 1 — 1 correspondence with the elements $\{X\}$ of L'. If X, Y have representatives $\{x(a) | \mathfrak{A}\}, \{y(b) | \mathfrak{B}\}$ we inflate them to phalanxes on the common stack $\mathfrak{C}$ and define $X + Y$ by the representative $\{x_c + y_c | \mathfrak{C}\}$ and kX by $\{kx(a) | \mathfrak{A}\}$. It is easily verified that these definitions are consistent with the correspondence of elements of L and the convergent Cauchy phalanxes and that L' is the (essentially unique) complete l. t. s. desired [5] when the neighborhood basis is defined by

$$V = \{X \mid \{x(a) + U_1 \mid \mathfrak{A}\} \subset U\}.$$

If $X, Y \epsilon V$ then, for some representative $\{x(a) \mid \mathfrak{A}\}$ and some $U_1 (U, U_1 \varepsilon \mathfrak{U}^s)$, some representatives, $\{x(a) + U_1 \mid \mathfrak{A}\}$, $\{y(b) + U_2 \mid \mathfrak{B}\}$ are in U. The convexity of $\mathfrak{U}^s$ implies $\lambda(x_c + U_3) + (1 - \lambda)(y_c + U_3) = \lambda x_c + 1 - \lambda y_c + U_3 \subset U$ where $U_3 \subset U_1 \cap U_2$ and hence $\lambda X + 1 - \lambda Y$, since it has the representative $\{\lambda x_c + 1 - \lambda y_c | \mathfrak{C}\}$, is contained in V.

An alternative argument of independent interest uses the quasi norms associated with a neighborhood basis. Let $|||X|||_{U^s} = \bar{L} \| \{x(a) | \mathfrak{A}\} \|_{U^s}$. It is easy to show that $|||X|||_{U^s}$ is independent of the choice of representative for X. Since $\| \|_{U^s}$ is triangular,

$$L \| \{x_c + y_c | \mathfrak{C}\} \|_{U^s} \leq \bar{L} \| \{x_c | \mathfrak{C}\} \|_{U^s} + \bar{L} \| \{y_c | \mathfrak{C}\} \|_{U^s}$$

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[5] Axioms I, II, IV reflect known properties of the completion of an Abelian topological group. Axiom III is easily verified. For V let $X \sim \{x(a) \mid \mathfrak{a}\}$. Suppose that V comes from U. Choose U_1 to satisfy $U_1 + U_1 + U_1 \subset U$. Then $x(a) \epsilon x(a_0) + U_1$, $a \succeq a_0$. For some $k \geq 1$, $x(a_0) \epsilon kU_1$. Then $x(a) + U_1 \subset kU_1 + U_1 + U_1 \subset kU$, $a \succeq a_0$ or $X \epsilon kV$, since $\mathfrak{a}' = \{a \mid a \succeq a_0\}$ can replace $\mathfrak{a}$.

whence it follows readily that $|||\ |||_{U^\alpha}$ is triangular. Order U^α by $U^\alpha \prec U^{\alpha'}$ if and only if $U^\alpha \subset U^{\alpha'}$. It may be verified that $\{|||\ |||_{U^\alpha} \mid U^\alpha \in \mathfrak{U}^\alpha\}$ is a collection of triangular quasi norms satisfying conditions [3b] guaranteeing that L' is a locally convex l. t. s. which is obviously complete and identical with that obtained in the preceding proof. Incidentally it is plain that the completion of a convex set is convex.

It would be of interest to investigate further the relations between convexity and regular L convexity. If every closed convex set is regularly L convex (this obviously implies $\Theta = \theta$), is L convex or is B bounded if $\hat{B}$ is bounded? It seems probable that both these implications are false.

LEMMA 12. *If K is closed and convex and $x_0 \bar{\in} K$ then there is a closed convex set K' with $K' \neq \emptyset$ such that $x_0 \in \bar{K}' \subset \bar{K}$.*

According to Lemma 5, $(K + \bar{N}_1) \cap (x_0 + \underline{N}_2) = \emptyset$. Since N_1 satisfies VI, $\bar{N}_1$ is convex [2] and therefore, in view of Lemmas 6 and 7, $\overline{K + \bar{N}_1}$ is convex.[6] Inasmuch as $x_0 + \underline{N}_2$ is open it follows that $(\overline{K + \bar{N}_1}) \cap + \underline{N}_2) = \emptyset$. Hence the conclusion of the lemma is valid for $K' = \overline{K + \bar{N}_1}$.

THEOREM 7. *If K is convex then K is $w(L)$ closed.*

This is an extension of a fundamental result for vector normed spaces due to Mazur [5]. If $K = L$ the theorem is trivially true. Suppose $x_0 \bar{\in} K$. The set K' of Lemma 12 is regularly L convex according to Theorem 1. Hence $F_0(x_0) > \sup_{K'} F_0(x) \geq \sup_{K} F_0(x)$ for some $F \in L^*$. That is to say K is regularly L convex, whence, by Theorem 4, K is $w(L)$ closed.

5. Metrizability of L^*. Let $B \subset L$ be a bounded set containing θ and let $\mathfrak{B}$ stand for the collection of all such sets. The neighborhoods B^*, of L^*, are defined by $B^* = \{f \mid |f(x)| < 1, x \in B\}$ and are plainly convex. The set B is said to *determine* B^*. The neighborhoods of f_0 are of the form $f_0 + B^*$. With this neighborhood topology L^* is an l. t. s. Plainly if $B' \supset B''$ then $B'^* \subset B''^*$. In general the converse implication is not valid, but cf. Lemma 14. We shall call the sequence $\{B_n\} \in \mathfrak{B}$ an *outer countable basis* if for every $B \in \mathfrak{B}$, we can satisfy $B \subset B_n$ for some n. (The notion of an outer countable

[6] Even if K_1 and K_2 contain inner points and are closed and convex, $K_1 + K_2$ need not be closed unless a compactness condition is imposed on K_1. For instance in the Euclidean plane let K_1 consist of the branch of the hyperbola $xy = 1$ in the first quadrant and all points lying above this branch. Let K_2 be the reflection of K_1 in the y axis. Then $K_1 + K_2$ consists of the upper half plane exclusive of $y = 0$ and accordingly is not closed.

basis and Theorem 8 were suggested to me in conversation by Professor Kakutani).

THEOREM 8. *If L has an outer countable basis $\{B_n\}$ then L^* is metrizable as an F space.*

It is well known that an l. t. s. space may be metrized as a possibly incomplete F space if there is a countable neighborhood basis at any point [3(b), 4]. Accordingly we need merely show there is a countable basis $\{B^*_n\}$ in L^*. Let the neighborhood B^* be arbitrary and let B be any determining set of B^*. By hypothesis $B_n \supset B$ for some n, and hence, $B^* \supset B^*_n$.

Simple examples show that the sufficiency condition stated in this theorem is not, generally, a necessary condition. If B is any determining set of B^* then $\{\lambda x \mid |\lambda| \leq 1, x \in B\}$ is obviously bounded and also a determining set of B^*. In the sequel *all determining sets are tacitly assumed to satisfy* $B = \{\lambda x \mid |\lambda| \leq 1, x \in B\}$.

LEMMA 13. *If L is locally convex and $B = -B$ then*

$$\bar{\hat{B}} = \{x \mid |f(x)| \leq 1, f \in B^*\}.$$

Denote $\{x \mid |f(x)| \leq 1, f \in B^*\}$ by B'. Now $\hat{B}$ and $\bar{\hat{B}}$ are bounded. Moreover according to Lemma 7, $\bar{\hat{B}}$ is convex. Write $D(= -D)$ for $\bar{\hat{B}}$ (this notation is used in the following theorem also). Now B' is clearly convex and closed and so $B' \supset D$. Also, D is regularly convex in view of Lemma 7 and Theorems 4 and 7. Hence for arbitrary $x_0 \bar{\in} D$ and some f_0,

$$|f_0(x_0)| = d > a = \sup_D |f_0(x)|.$$

Write f_1 for $2f_0/a + d$. Then

$$|f_1(x_0)| > 1 > \sup_D |f_1(x)|.$$

Accordingly $f_1 \in D^* \subset B^*$ and therefore $x_0 \bar{\in} B'$. Thus $D \supset B'$, and so $D = B'$. Evidently we have as a special case

$$\bar{\hat{B}} = \{x \mid |f(x)| \leq 1, f \in D^*\} = \{x \mid |f(x)| \leq 1, f \in (\hat{B})^*\}.$$

THEOREM 9. *If L is locally convex then a necessary condition for L^* to be metrizable is that L admit an outer countable basis.*

If L^* is metrizable there is a countable neighborhood basis $\{B^*_n\}$ at θ^*. Denote by $B_n \in B$ any determining set of B^*_n. Define D_n as $\bar{\hat{B}}_n$. Cf. Lemma 13. We wish to show that $\{D_n\}$ is an outer countable basis. If this were not true,

then, for some $B' \varepsilon B$, we should have $B' \frown \bar{D}_n \neq \emptyset$ for all n. Since $D^*_n \subset B^*_n$ it is clear that $D^*_m \subset B'^*$, where m is some fixed value of n. Since D_m is regularly L convex, if $x_0 \varepsilon B' \frown \bar{D}_m$ we have

$$\sup_{D_m} F_0(x) = -\inf F_0(x) = a \geq 0 \quad \text{and} \quad F_0(x_0) - a = \delta > 0.$$

Then, with $F_1 = 2F_0/2a + \delta$, we have $F_1(x_0) > 1 > \sup_{D_m} |F_1(x)|$. That is to say $F_1 \varepsilon D^*_m$ and $F_1 \bar{\varepsilon} B'^*$ or $D^*_m \not\subset B^{*\prime}$ in contradiction with the definition of D^*_m. Accordingly $B' \frown \bar{D}_m = \emptyset$ and hence $\{D_n\}$ is an outer countable set.

LEMMA 14. *If L is locally convex and $B^*_1 \supset B^*_2$ then, if B_i is a determining set of B^*_i, $\bar{\bar{B}}_1 \subset \bar{\bar{B}}_2$.*

The boundedness and convexity of $\bar{\bar{B}}_i$ are covered in the proof of Theorem 9. By Lemma 13,

$$\bar{\bar{B}}_1 = \{x \mid |f(x)| \leq 1, f \varepsilon B^*_1\} \subset \{x \mid |f(x)| \leq 1, f \varepsilon B^*_2\} = \bar{\bar{B}}_2.$$

The following theorem extends an important result due to Helly [10].

THEOREM 10. *Let L be locally convex and L^* metrizable. Then there is a fixed closed convex bounded set B in L with the property that, whenever $\{f^i\} \subset L^*$ and $\{c_i\}$, $i = 1, \cdots, N$ are so chosen that*

$$|\Sigma_1^N \lambda_i c_i| \leq (M(\Sigma_1^N \lambda_i f^i), \theta^*)$$

for arbitrary $\{\lambda_i\}$, the system $f^i(x) = c_i$ is satisfied by some $x_\epsilon \varepsilon (M + \epsilon)B$ for each $\epsilon > 0$.

By a well known result of Eidelheit and Mazur [11] on equivalent metrics in an F space, we may require that $(\mu f, \theta^*)$ be strictly monotone in $\mu \geq 0$ for each fixed f (the demonstration given by the authors remains valid when the completeness requirement on F is dropped). Moreover $(f, \theta^*) = (-f, \theta^*)$ [8]. For some set in the outer countable basis of L, say

$$B = \bar{\bar{B}}, \{f \mid (f, \theta^*) < 1\} \supset B^*.$$

Denote the general point $\xi_1, \cdots, \xi_N$ in the N dimensional Euclidean space by z. Let

$$K = -K = \{z \mid \xi_i = f^i(x), x \varepsilon (M + \epsilon)B\}.$$

Evidently K is a convex, bounded, continuous map of $(M + \epsilon)B$. If the theorem were untrue $z_0 = (c_1, \cdots, c_N) \bar{\varepsilon} K$. Then some supporting hyperplane, $\Sigma_i^N \lambda_i \xi^i = 1$, of K contains z. Thus

$$|\Sigma_1^N \lambda_i f^i(x)| \leq 1 \quad \text{for} \quad x \in (M+\epsilon)B$$

while

$$\Sigma_1^N \lambda_i c_i = 1 \quad \text{or} \quad |\Sigma \lambda_i f^i(x)| < 1 \quad \text{for} \quad x \in (M+\epsilon/2)B,$$

i. e., $\Sigma \lambda_i f^i \in (M+\epsilon/2)B^*$. Accordingly,

$$((M+\epsilon/2)\Sigma_1^N \lambda_i f^i, \theta^*) < 1 = |\Sigma_1^N \lambda_i c_i|.$$

Then

$$(M\Sigma \lambda_i f^i, \theta^*) < 1 = |\Sigma_1^N \lambda_i c_i|.$$

This is in contradiction with the hypothesis of the theorem.

If the metric satisfied $(\mu f, \theta^*) \leq \mu(f, \theta^*)$, for all f and $\mu \geq 0$, we could of course replace $(M(\Sigma_1^N \lambda_i f^i), \theta^*)$ by $M(\Sigma_1^N \lambda_i f^i, \theta^*)$ but such a metric must obvously be a norm. (Take $\mu' = \mu^{-1} > 0$ and $f' = \mu f$, then it follows that $\mu'(f', \theta^*) \leq (\mu' f', \theta^*) \leq \mu'(f', \theta^*)$). If B is $w(L)$ bicompact or bicompact of order $\aleph_a$, Theorem 10 may be extended to the case that $\mathfrak{a}$ in $\{c_a \mid \mathfrak{a}\}$ has power at most $\aleph_a$, by straightforward modification of the writer's proof for the Banach space case [Theorem 15; 12].

THEOREM 11. *If L is locally convex with L^* metrizable and separable then L is separable.*

The hypotheses of the theorem imply the existence of an outer countable basis $\{B_n\}$. Evidently the neighborhood systems

$$\{f \mid |f(x)| < \epsilon, x \in B_n\} \quad \text{and} \quad \{f \mid |f(x)| < 1, x \in B_n\}$$

are equivalent. Since $\cup B_n = L$, if the theorem were false some B_n, denoted henceforth by B, would not be separable. Let Γ be the set $\{f \mid \|f\|_B = \frac{1}{2}\}$ where $\|f\|_B = \sup_B |f(x)|$. In view of the separability of L^* a sequence $\{f^\sigma\} \subset \Gamma$ may be found which satisfies the condition

$$\{f^\sigma\} \frown \{f \mid \|f_0 - f\|_B < \epsilon\} \neq \emptyset;$$

for any $\epsilon > 0$ and $f_0 \in \Gamma$. Since $f^\sigma \in \Gamma$, $|f^\sigma(x^\sigma)| \geq \delta$ for some $x^\sigma \in B$ and $0 < \delta < \frac{1}{2}$. Let L_B and $L_{\{\sigma\}}$ be the closed linear manifolds determined by B and $\{x^\sigma\}$ respectively. Since $L_{\{\sigma\}}$ is separable some x, say x_0, is in L_B but not in $L_{\{\sigma\}}$. Now $L_{\{\sigma\}}$ is convex and closed and by Theorem 7 is a regularly L convex linear manifold. Hence for some

$$f_1 \in L^*, f_1(x_0) > \sup_{L_{[\sigma]}} f_1(x) = 0.$$

Moreover, since f_1 is linear, Theorem 2 of Wehausen guarantees that $\| f_1 \|_B < \infty$. Hence $f_0 = f_1/2 \| f_1 \|_B \varepsilon \Gamma$. Now

$$\| f_0 - f^\sigma \|_B \geq | f_0(x^\sigma) - f^\sigma(x^\sigma) | \geq \delta$$

for all σ which is in contradiction with the definition of $\{f^\sigma\}$.

The theorem is still true if the condition of metrizability of L^* is replaced by the requirement that there be a denumerable collection $\{C_n\}$ of bounded sets in L such that [7]

$$\text{(a)} \quad \bigcup_{m=1}^{\infty} \bigcup_{n=1}^{\infty} mC_n = L \quad \text{and} \quad \text{(b)} \quad B \subset \bigcup_{m=1}^{\infty} mC_n$$

for each bounded set B and some n.

THEOREM 12. *If L is locally convex and L^* is normable then* $\bigcup_{n=1}^{\infty} nB = L$ *for some bounded total set B in L.*[7]

Define B as in Theorem 10. Let x' be arbitrary and write B' for $\{tx' \mid | t | \leq 1\}$. Then $B^* \supset \{\beta f \mid \| f \| < 1\} \supset \beta B^*$ for some $\beta > 0$. In view of Lemma 14, $\beta B \supset B'$; that is to say $x' \varepsilon mB$ for any integer $m > \beta$ and hence $\cup_n B = L$. Evidently $f(x) = 0$ for all $x \varepsilon B$ implies $f = \theta^*$. The theorem may also be established by considering the topology induced on L by L^{**} and clearly B may be taken as $\{x \mid | f(x) | \leq 1, \| f \| \leq 1\}$.

COROLLARY. *If L is locally convex and of the second category then a necessary and sufficient condition for L to be normable is that L^* be normable.*

The condition is obviously necessary. On the other hand if L contained no bounded neighborhood then B would be nowhere dense and hence L would be of the first category. However the presence of a bounded neighborhood means that L is both locally convex and locally bounded and hence normable.

The following example is of interest here and in connection with Theorem 19 below. Let E be a non-separable, reflexive Banach space. We denote by L the elements of E under the $w(E)$ topology. Then L is a sequentially complete, non-metrizable, locally convex space of the first category. Nevertheless L^* is normed. Incidentally it is apparent that if $\bar{L} \| f^n(x) \| < \infty$ for all $x \varepsilon L$, then $\| f^n \| < M$ (in particular $w^*(L^*)$ convergence of $\{f^n\}$ to f_0 implies $\| f^n \| < M$) yet L is of the first category.

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[7] It is not true however that if $L = U_{m=1}^{\infty} mB$, then $\{mB\}$ is an outer countable basis, even if L is normed. The Hilbert parallelotope, taken as the set B, in L^2, furnishes an immediate counter example.

6. Local boundedness. Let A be a bounded open set in L and let

$$A' = \{\lambda x \mid |\lambda| \leq 1,\ x \in A\}.$$

Then A' is easily verified to be bounded and open in L. Define the quasi norm $\|x\|$ by the usual scheme

$$\|x\| = \inf\{h \mid x \in hA'\}.$$

Hyers [3a] has shown that this quasi norm satisfies (1) $\|x\| \geq 1$ and $\|x\| = 0$ implies $x = \theta$. (2) $\|tx\| = |t|\,\|x\|$. (3) For every $\eta > 0$ there is a $\delta > 0$ such that $\|x + y\| < \eta$ whenever $\|x\| < \delta$ and $\|y\| < \delta$. (4) The sets $\{x \mid \|x\| < \epsilon, \epsilon > 0\}$ form a system equivalent to $\mathfrak{U}$. (5) For every $x \in L$ and $\eta > 0$ there is a $\delta > 0$ such that $\|y\| - \|x\| < \eta$ for $\|y - x\| < \delta$.

We shall extend and sharpen these results significantly.

THEOREM 13. *If L is locally bounded the quasi norm satisfies*

$$\|x_1 + x_2\| \leq b(\|x_1\| + \|x_2\|)$$

for some $b \geq 1$ depending on A and L.

We shall refer to b as the *multiplier* of the quasi norm. The neighborhood system at θ may be taken as $\mathfrak{U} = \{U \mid U = A'/n,\ n = 1, \cdots\}$. For some values of p we have $A' + A'/2 \subset pA'$, $p \geq 1$, by IV. Hence p' exists such that $p'A' \supset tA' + (1-t)A'$ for all $0 \leq t \leq 1$. Let b be the inf of all such p'. If $\|x_i\| = \lambda_i$, $i = 1, 2$, then $x_i \in (\lambda_i + \epsilon)A'$ for all $\epsilon > 0$. Therefore

$$x_1 + x_2 \in (\lambda_1 + \lambda_2 + 2\epsilon)\left[\frac{\lambda_1 + \epsilon}{\lambda_1 + \lambda_2 + 2\epsilon}\,A' + \frac{\lambda_2 + \epsilon}{\lambda_1 + \lambda_2 + 2\epsilon}\,A'\right] \subset (b + \delta)(\lambda_1 + \lambda_2 + 2\epsilon)A'.$$

Since this is true for each $\delta, \epsilon > 0$ it follows that

$$\|x_1 + x_2\| \leq b(\lambda_1 + \lambda_2) = b(\|x_1\| + \|x_2\|).$$

It thus appears that the quasi norm for a locally bounded L is formally similar to that for the spaces L_p, $0 < p < 1$. (The example of the spaces l_p, $0 < p < 1$ shows that functionals exist even when the norm is not triangular). It is important to remark that b should be written more felicitously b_A for, as is easily seen, it varies in general with the choice of the bounded open set, A. The quantity $\beta_L = \inf\{b_A \mid A$ bounded and open in $L\}$ is a characteristic of L alone. However β_L may not be a multiplier. Cf. Section **7**. All references to b below are really to a b_A.

Let A_i be a bounded open set in L_i, $i = 1, 2$, and associate A'_i as before. Let P be a linear operator with domain L_1 and range in L_2. The subscript $i = 1, 2$ affixed to a quasi norm or to b will indicate the space under consideration. Define $\| P \|$ by $\| P \| = \sup_{A'_1} \| P(x) \|_2$. The space of such operators is denoted by $\underline{\boldsymbol{P}}$ and it is known that $\underline{\boldsymbol{P}}$ is a locally bounded l. t. s. satisfying (1), (2) and (5), [3(a)].

THEOREM 14. *The quasi norm in* $\boldsymbol{P}$ *satisfies*

$$\| P_1 + P_2 \| \leq b_2(\| P_1 \| + \| P_2 \|).$$

Evidently

$$\begin{aligned} \| P_1 + P_2 \| &= \sup_{A'_1} \| P_1(x) + P_2(x) \|_2 \\ &\leq \sup_{A'_1} b_2(\| P_1(x) \|_2 + \| P_2(x) \|_2) \\ &\leq b_2(\| P_1 \| + \| P_2 \|). \end{aligned}$$

LEMMA 15. *If* L_2 *is normed,* (a) $\underline{\boldsymbol{P}}$ *is a normed vector space and* (b) *If* P *is additive and* $\| P(x) \| \leq M \| x \|$ *for all* $x \in L_1$, *then* P *is linear* [3a].

Condition (a) is satisfied for instance when $\underline{\boldsymbol{P}}$ is L^*.

THEOREM 15. *The locally bounded space* L *may be completed with preservation of local boundedness.*

Let the equivalent metric neighborhoods of L (cf. Section **5**) be written $S_d = \{x \mid (x, \theta) < d\}$. Consider the typical element of L' to be the class of equivalent Cauchy sequences in L, where two Cauchy sequences are equivalent if they 'mesh' to a Cauchy sequence. The points of L' are denoted by capitals and the distance is defined by $(X, Y) = \lim (x^n, y^n)$ where $X \sim \{x^n\}$, $Y \sim \{y^n\}$ i. e., $\{x^n\}$ and $\{y^n\}$ are representative Cauchy sequences of the classes defining X and Y respectively. As is well known this distance is a metric and if $\{x^n\}$ converges to x_0 and $\{y^n\}$ to y_0, then $(X, Y) = (x_0, y_0)$. Moreover in view of the triangle property of the metric, $S_0 = \{X \mid (X, \theta) < \delta\}$ is an open set. Plainly L' is an l. t. s. and is complete in the metric sense. Now A contains s_D, where D is some positive d, by the equivalence of the original and metric topologies. Then s_D is bounded in L and so S_D is bounded in L'. Indeed this is tantamount to the assertion that for arbitrary $e < D$ and some h we have $hS_D \subset S_e$. Let h' satisfy $h's_D \subset s_{e'}$, $e' < e$. Then if $X \in S_D$ and $X \sim \{x^n\}$ we must have $x^n \in s_D$ for $n > N$. Hence $(h'x^n, \theta) < e'$

for $n > N$. Thus $(h'X, \theta) = \lim (h'x^n, \theta) < e$. Since S_D is bounded and open, L' is locally bounded. Incidentally $s_D = \{\lambda x \mid x \in s_D, \mid \lambda \mid \leqq 1\}$ and $S_D = \{\lambda X \mid \mid \lambda \mid \leq 1, X \in S_D\}$.

THEOREM 16. *If L is locally bounded and x^n converges to x_0 then*

$$\| x_0 \| \geq \bar{L} \| x^n \| \geqq \underline{L} \| x^n \| \geq \| x_0 \|/b.$$

We have $\| x^n - x_0 \| \to 0$. From Theorem 13 we have

$$b \| x^n \| \geq \| x_0 \| - b \| x^n - x_0 \| \quad \text{or} \quad \underline{L} \| x_n \| \geq \| x_0 \|/b.$$

On the other hand, in consequence of (5), $\| x_0 \| > \| x^n \| - \eta$ for $n > N_\eta$. Hence $\bar{L} \| x^n \| \leq \| x_0 \|$.

A consequence is that the quasi norm relative to A' is always upper semi-continuous but not necessarily continuous except at θ and, in fact, Hyers has given an elementary example of a discontinuous quasi norm. These remarks lend interest to the next result.

THEOREM 17. *If L is locally bounded there is a continuous quasi norm.*[8]

Let s_δ, defined above, be the basic open bounded set defining the quasi norm. Let $x^n \to x_0$ and denote $\| x^n \|$ by λ_n. We need only the case $\lambda_0 \neq 0$. We assume without loss of generality that $(\mu x, \theta)$ is monotonic increasing in μ, $\mu \geq 0$ [11]. Now, for $0 < \delta < \lambda_0$,

$$\begin{aligned}(x_0/(\lambda_0 - \delta), \theta) > d &> (x^n/(\lambda^n + \delta), \theta) \\ &\geq (x_0/(\lambda^n + \delta), \theta) - (x^n - x_0)/(\lambda^n + \delta), \theta).\end{aligned}$$

Thus

$$(x_0/(\lambda_0 - \delta), \theta) \geq \bar{L}(x_0/(\lambda^n + \delta), \theta).$$

Hence

$$(\lambda_0 - \delta)^{-1} \geq \bar{L}(\lambda^n + \delta)^{-1}.$$

Thus $\underline{L}\lambda_n \geq \lambda_0 - 2\delta$ or, since $\delta > 0$ is arbitrary, $\underline{L} \| x_n \| \geqq \| x_0 \|$. In view of Theorem 16,

$$\| x_0 \| \geq \bar{L} \| x^n \| \geq \underline{L} \| x^n \| \geq \| x_0 \|,$$

or $\| x \|$ is continuous. (We have shown incidentally that a certain sphere in the equivalent metric is the A required).

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[8] This gives an affirmative answer to the "general question" raised by Hyers [3(a), p. 566].

7. An example. This section concerns itself with an example of a locally bounded l. t. s. for which β_L is not a possible multiplier. Consider the linear sequence space $\mathfrak{L}$ with the distance given by

$$(x, y) = \sum_{n=1}^{\infty} | x_i - y_i |^{p_i}$$

where

$$p_i = (1 + (\log (i+1))^{-1/2})^{-1}.$$

Write $q_i = p_i^{-1}$. Evidently p_i increases monotonely to 1. It is easy to verify that (x, y) is a metric and that $\mathfrak{L}$ is an l. t. s. Since $(\lambda x, \theta) \leq \lambda^{p_i}(x, \theta)$ for $0 \leq \lambda \leq 1$, it follows that $\mathfrak{L}$ is locally bounded. Although it is irrelevant to our purpose, it may be mentioned that the standard procedure establishes that $\mathfrak{L}$ is complete. It is plain that an equivalent metric, denoted by $(x, y)_N$, is defined by

$$(x, y)_N = \sum_{i=1}^{N} | x_i - y_i |^{p_N} + \sum_{N+1}^{\infty} | x_j - y_j |^{p_j}.$$

It is an easy matter to show that, with the choice $A' = \{x \,|\, (x, \theta)_N < 1\}$, the associated multiplier b is $2^{q_N - 1}$. Hence $\beta_{\mathfrak{L}} = 1$. We assert that 1 *is not a possible multiplier.* Suppose the assertion false. Then $\mathfrak{L}$ is normable and hence locally convex. Then the neighborhood $s = \{x \,|\, (x, \theta) < 1\}$ contains a convex open set K which includes $s_\epsilon = \{x \,|\, (x, \theta) < \epsilon\}$ for some $\epsilon > 0$. Hence $s_\epsilon \subset K \subset s$. Now s_ϵ contains

$$x^n = (\epsilon^{q_1}\delta_1{}^n, \cdots, \epsilon^{q_i}\delta_i{}^n, \cdots)$$

where $\delta_i{}^n$ is the Kronecker delta. Then

$$z^M = \Sigma_1{}^M \alpha_n x^n / \Sigma_1{}^M \alpha_n \in s_\epsilon$$

where $\alpha_n = (n+1)^{-q_n}$. Manifestly $\alpha_n \downarrow 0$ and hence the convergence of $\Sigma_1{}^\infty \alpha_n$ is guaranteed by the fact that

$$\int_1^\infty \frac{dx}{x^{1+(\log x)^{-1/2}}} = \int_0^\infty e^{-w^{1/2}}\, dw \text{ converges.}$$

Write $\sigma = \Sigma_1{}^\infty \alpha_n$ where manifestly $\sigma > 1$. Then

$$(z^M, \theta) > (\Sigma_1{}^M \alpha_n x^n / \sigma, \theta)$$
$$= \epsilon \Sigma_1{}^M (1/n+1)\sigma^{-p_n} > (\epsilon/\sigma)\Sigma_1{}^M (n+1)^{-1} \to \infty,$$

which is incompatible with $s_\epsilon \subset s$. Actually all that is needed for this example

is $p_i \uparrow 1$. It may be remarked parenthetically that if $p_i \downarrow 0$, the resulting space is neither locally bounded nor locally convex.

8. The conjugate space L^*. The $w^*(L^*)$ topology is induced by the neighborhood system $\mathfrak{U}^* = \{U^*\}$ where

$$U^* = U^*(\theta^*; x_1, \cdots, x_n; \epsilon) = \{f \mid |f(x_i)| < \epsilon;\ i = 1, \cdots, n\}.$$

Obviously X_i of Section **3** may replace x_i and L^* is an l. t. s. in its $w^*(L^*)$ topology. We define *regular L^* convexity* for $K \subset L^*$ by the property that if $f_0 \bar{\varepsilon} K$ then there is an $x_0 \varepsilon L$ for which $f_0(x_0) > \sup_K f(x_0)$. This implies that K is convex.

THEOREM 18. *The statements* (a) *$K \subset L^*$ is regularly L^* convex and* (b) *K is convex and $w^*(L^*)$ closed, are equivalent.*

The proof follows the pattern of the demonstration of Theorem 4 with the roles of L and L^* interchanged.

LEMMA 16. *If L is locally bounded the system*

$$\mathfrak{U}' = \{U'\}, \qquad U' = \{f \mid f(x_i)| < \epsilon, x_i \varepsilon A'\}$$

is equivalent to $\mathfrak{U}^*$.

The lemma is a trivial consequence of the fact that A' is total [8, p. 58] for L^*, i. e., $f(x) = 0$ for all $x \varepsilon A'$ requires $f = \theta^*$.

THEOREM 19. *If L' is the completion of L then L^* and L'^* are quasi equivalent.*

Quasi equivalence is defined below. Sets and functions in L' are primed and the theorem stems from the fact that with our method of completion L is isomorphic to a dense set in L'. In this sense $L \subset L'$. Since f is uniformly continuous on L it has a unique extension f'. Moreover every f' determines a unique f. Thus $(A) L^*$ and L'^* are in $1-1$ correspondence. Using Zorn's lemma we can define a maximal family $\mathfrak{D}' \subset \mathfrak{B}'$ (cf. Section **5**) subject to (a) $B' = \{\lambda X \mid |\lambda| \leq 1, X \varepsilon B'\}$, $B' \varepsilon D'$. (b) No set of $\mathfrak{D}'$ is dense in another. Write $B = B' \cap L$. The sets B comprise a maximal family D with properties analogous to (a) and (b). Plainly $(B)\mathfrak{D}$ and $\mathfrak{D}'$ are in $1-1$ correspondence. If the sets of $\mathfrak{D}'(\mathfrak{D})$ are ordered by inclusion the quasi norms $\|f'\|_{B'}$ ($\|f\|_B$) are triangular and satisfy the conditions of Hyers [3b]. They determine the ordinary topology of L^* and L'^*. If B corresponds to B', B is dense in B' so (C) $\|f\|_B = \|f'\|_{B'}$. The satisfaction of (A), (B), (C) con-

stitutes quasi equivalence. For a locally bounded L, L^* and L'^* are easily shown to be complete and the quasi equivalence reduces to Banach equivalence [8].

Many results concerned with the w^* topology of the conjugate space of a vector normed space have valid analogues when L is a locally bounded l. t. s. The following results are of this type where, to avoid triviality, we assume throughout that L^* *contains more than one point.*

THEOREM 20. *If L is locally bounded then necessary and sufficient conditions for the $w^*(L^*)$ sequential convergence of $\{f^n\}$ to f_0 are* (a) $\| f^n \| \leq M$ and (b) $\{f^n(x)\}$ *converges pointwise on a dense set in L.*

THEOREM 21. *If L is separable then S^* is $w^*(L^*)$ sequentially separable.*

THEOREM 22. *If L is locally bounded and S^* is the unit sphere $\{f \mid \| f \| \leq 1\}$ then S^* is bicompact in the $w^*(L^*)$ topology.*

We vary the Banach space type of proof [13] by not making use of the Tychonoff theorem. It is easy to show that S^* is $w^*(L^*)$ complete. Moreover it is immediate that S^* is totally bounded or what is equivalent, largely compact [7; p. 64], in the sense that for every $W(\theta^*; x^1, \cdots, x^r; \epsilon)$ and some finite collection $f^1, \cdots, f^m$ in S^*, $S^* \subset \bigcup_{j=1}^{m} W(f^j; x^1, \cdots, x^r; \epsilon)$. Accordingly S^* is $w^*(L^*)$ bicompact [6, 7, Theorem 8.14]. Similarly if $\theta = \Theta$, $w(L)$ completeness of a bounded set implies $w(L)$ bicompactness.

The definition of transfinite limits given by Banach [8] may be used without change for L^* when L is locally bounded.

LEMMA 17. *For L locally bounded, if*

$$\| f_\xi \| \leq M, \qquad 1 \leq \xi < \tau, \tau$$

a limit ordinal, and $f_\xi \in L^*$, then $f_0 \in L^*$ exists, such that

$$\underset{\xi<\tau}{L} f_\xi(x) \leq f_0(x) \leq \underset{\xi<\tau}{\bar{L}} f_\xi(x)$$

for all $x \in L$ and $\| f_0 \| \leq M$.

Let $p(x) = \bar{L} f_\xi(x)$ in the notation of Banach, p. 28. Then $p(x) \leq M \| x \|$ and $p(x)$ satisfies the conditions of the corollary to the Hahn-Banach Theorem (though $M \| x \|$ cannot be used for the same purpose since

the quasi norm is not triangular). Hence an additive functional f_0 exists satisfying $|f_0(x)| \leq M \|x\|$ and therefore, by Lemma 15(b) f_0 is linear. The function f_0 is a *transfinite limit* of $\{f_\xi\}$. A set $K \subset L^*$ is transitively closed if it contains all its transfinite limits and plainly it is then convex.

LEMMA 18. *For L locally bounded and $\|f_\xi\| \leq M$, $1 \leq \xi < \tau$, if f_1 satisfies*

$$\underset{\xi<\tau}{\underline{L}} f_\xi(x) \leq f_1(x) \leqq \underset{\xi<\tau}{\bar{L}} f_\xi(x)$$

for $x \in Q \subset L$ then there is a transfinite limit f_0, $\|f_0\| \leq M$, of $\{f_\xi\}$ which agrees with $f_1(x)$ on the closed linear extension of Q.

LEMMA 19. *If L is locally bounded, and if $K \subset L^*$ is either $w^*(L^*)$ closed, regularly L^* convex or transfinitely closed, then K is closed.*

THEOREM 23. *If L is locally bounded and $K \subset L^*$ is bounded then a necessary and sufficient condition that K be $w^*(L^*)$ closed and convex is that K is transfinitely closed.*

The necessity argument may be patterned with obvious changes on Banach, p. 122, paragraph 2, and Theorem 20.

Let $\bar{K}$ be the $w^*(L^*)$ closure of K. Suppose $\bar{K} \neq K$ and let $f_0 \in \bar{K} \cap \tilde{K}$. Well order all the sets, X, of L, i. e., $\{X_\xi \mid 1 \leq \xi < \gamma\}$ where γ is a limit number (since $L^* \neq \theta^*$). We develop a familiar type of transfinite induction argument. The fact that $\|f\| \leqq M$ for all K is tacitly assumed in all statements made below about the existence of various transfinite limits. Since $f_0 \in \bar{K}$, for any X_η and integer m there is a linear functional in K, say $f_\eta{}^m$ which satisfies

$$|f_\eta{}^{(m)}(X_\eta) - f_0(X_\eta)| < 1/m.$$

Hence

$$\underset{m}{\underline{L}} f_\eta{}^m(X_\eta) - 1/m = \underset{m}{\underline{L}} f_\eta{}^m(X_\eta) \leq f_0(X_\eta)$$

$$\leq \underset{m}{\bar{L}} f_\eta{}^m(X_\eta) + 1/m = \underset{m}{\bar{L}} f_\eta{}^m(X_\eta).$$

In view of the transfinite closure of K and Lemma 18, some transfinite limit,

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[9] If, for instance, K is a closed vector subspace of L such an analysis as that given by Banach for the case L is a Banach space meets difficulties in the last step due to the fact that $b > 1$, [8, p. 119].

$f_\eta \in K$, of $\{f_\eta^m\}$ exists satisfying $f_\eta(X_\eta) = f_0(X_\eta)$. Suppose η is not a limit ordinal and that $f^{\eta-1} \in K$ satisfies $f^{\eta-1}(X_\rho) = f_0(X_\rho)$ for $\{X_\rho \mid 1 \leq \rho < \eta - 1\}$. We have shown the existence of an element $f_{\eta-1}$ in K such that $f_{\eta-1}(X_{\eta-1}) = f_0(X_{\eta-1})$. Define $\{F^\nu \mid 1 \leq \nu < \eta - 1\}$ by $F^\nu = f^{\eta-1}$, $\nu < \eta - 1$ and $F^{\eta-1} = f_{\eta-1}$. The transfinite limit, denoted by f^η, agrees with f_0 on $\{X_\rho \mid 1 \leq \rho < \eta\}$.

Suppose that for some $\eta \leq \gamma$ it is impossible to find an element of K which agrees with f_0 on $\{X_\rho \mid 1 \leq \rho < \eta\}$. Let Q be the set of such η's. This set has a first element η_0 and in view of considerations given above we need only consider the case that η_0 is a limit number. Then for every $\zeta < \eta_0$ there is an f^ζ such that $f^\zeta(X_\rho) = f_0(X_\rho)$ for $1 \leq \rho < \zeta$. Hence

$$\underset{\xi < \eta_0}{L} f^\xi(X_\rho) \leq f_0(X_\rho) \leq \bar{L} f^\xi(X_\rho) \quad \text{for} \quad 1 \leq \rho < \eta_0.$$

Accordingly an $f' \in K$ agrees with f_0 on $\{X_\rho \mid 1 \leq \rho < \eta_0\}$ or $\eta_0 \bar{\in} Q$. Hence $Q = \emptyset$ and $f_0 \in K$ or $\bar{K} \cap \tilde{K} = \emptyset$. The writer has not been able as yet to settle the question of whether the theorem remains valid if the boundedness restriction on K is dropped.

THEOREM 24. *If L is locally bounded and $\Gamma \subset L^*$ is compact then Γ is $w^*(L^*)$ bicompact.*

By reason of earlier theorems in this section the proof follows the pattern of the Banach space case [12].

In view of the existence of a Hamel base every l. t. s. admits non-null distributive functionals [8; p. 231]. Let D^* be the space of distributive functionals on L under the $w^*(D^*)$ topology defined in the obvious way. It is trivial that if L is finite dimensional then $D^* = L^*$.

THEOREM 25. *If $\Theta = \theta$ then L^* is $w^*(D^*)$ dense in D^*.*

This is an obvious consequence of the argument in the first part of the corollary to Theorem 4.

UNIVERSITY OF ILLINOIS,
URBANA, ILLINOIS.

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TWO OBSERVATIONS CONCERNING ALGEBRAIC EQUATIONS FOR MATRICES.*

By IVAN NIVEN.

1. Let W be a matrix of order n in a field k, and $\psi(z) = 0$ its minimum equation in k of degree $m = m(W) \leqq n$. This integer m is called the index of W. Any polynomial $F(z)$ in k defines a matrix $A = F(W)$, and all matrices of this form constitute a ring $R[W]$ of order m over k; indeed $F(W) = 0$ if and only if $F(z)$ is divisible by $\psi(z)$. Hence $m(A) \leqq m(W)$, and in case $m(A) = m(W)$, the ring $R[A]$ coincides with the ring $R[W]$, so that W is a polynomial in A with coefficients from k. A matrix A is said to be non-derogatory if its characteristic equation is its minimum equation, $m(A) = n$. Now suppose that the matrix A is given, and that W is to be found. Then we have the following theorem:

If a polynomial $F(z)$ and a matrix A in a field k are given, then a solution W of $F(W) = A$ is expressible as a polynomial in A with coefficients from k, $W = f(A)$, if and only if the index of W equals the index of A. In case A is non-derogatory, every solution W of $F(W) = A$ is a polynomial in A, and is likewise non-derogatory.

From now on we assume k to be algebraically closed, so that the minimum function $\psi(z)$ of the given matrix A in k splits into powers of r distinct linear factors $z - \alpha_i$,

$$\psi(z) = \prod_{i=1}^{r} (z - \alpha_i)^{m_i}.$$

2. Let $P(w, z)$ be a polynomial of two variables w, z in k which actually contains w; we wish to solve the equation $P(W, A) = 0$ by a matrix $W = f(A)$ in $R[A]$. Considering the equation $P(w, z) = 0$ as defining an algebraic function $w(z)$, we speak of the finite Taylor expansion

$$\phi(z) = \beta + \beta_1(z - \alpha) + \cdots + \beta_{h-1}(z - \alpha)^{h-1} \tag{1}$$

* Received May 4, 1942.

as a branch at $z = \alpha$ which is regular up to the h-th order if

$$P(\phi(z), z) \equiv 0 \pmod{(z-\alpha)^h}.$$

The condition $P(f(A), A) = 0$ amounts to

$$P(f(z), z) \equiv 0 \pmod{\psi(z)},$$

or to the set of r simultaneous congruences

$$P(f(z), z) \equiv 0 \pmod{(z-\alpha_i)^{m_i}} \qquad (i = 1, \cdots, r),$$

whence follows this almost trivial

Principle of Construction.[1] *If for each* $i = 1, \cdots, r$ *we can obtain a branch* $\phi_i(z)$ *at* $z = \alpha_i$ *which is regular up to the order* m_i *then the simultaneous congruences*

$$f(z) \equiv \phi_i(z) \pmod{(z-\alpha_i)^{m_i}} \qquad (i = 1, \cdots, r)$$

uniquely determine a solution $W = f(A)$ *of our equation* $P = 0$. *Every solution* $W \in R[A]$ *is determined in this fashion.*

Special cases:

a) For $\alpha = \alpha_i$ choose a root $w = \beta$ of the equation $P(w, \alpha) = 0$. If

$$\frac{\partial P(\beta, \alpha)}{\partial w} \neq 0 \tag{2}$$

we get a uniquely determined regular branch $\phi(z)$ at $z = \alpha$ which begins with the constant term β and may be continued up to an arbitrarily high order.

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[1] For the general formulation of this principle of which I had proved a special case, I am indebted to Professors MacDuffee and Weyl. The principle is quite comprehensive; our special case (b), for example, embraces the main results in W. E. Roth's paper, "A solution of the matrix equation $P(X) = A$," *Transactions of the American Mathematical Society*, vol. 30 (1928), pp. 579-596. A different approach to the problem is to be found in the first half of M. H. Ingraham's paper, "Rational methods in matrix equations," *Bulletin of the American Mathematical Society*, vol. 47 (1941), pp. 61-70.

b) An equation of the type $F(w) = z$. The condition (2), $F'(\beta) \neq 0$, is not only sufficient, but also necessary for the regular expansion $\phi(z)$ at $z = \alpha$ to permit continuation beyond the initial term β. Combining this with the second conclusion in **1**, we obtain the following result: Under the hypothesis that A is non-derogatory, the equation $F(W) = A$ is solvable if and only if for every *multiple* root α of the minimum equation $\psi(z) = 0$ of A the equation $F(w) - \alpha = 0$ has at least one *simple* root $w = \beta$. If for every $i = 1, \cdots, r$ we choose β_i as a root of $F(w) - \alpha_i = 0$, which is simple in case α_i is a multiple root of $\psi(z) = 0$, then there is exactly one solution $W = f(A)$ satisfying the r conditions $f(\alpha_i) = \beta_i$, and all solutions W are obtained in this fashion.

c) The equation $w^q = z$, q a positive integer. By case (b) the matrix equation $W^q = A$ has a solution which is a polynomial in A if and only if 0 is not a multiple root of the minimum equation of A. The polynomial (1) is seen to have coefficients given by the formulas

$$\beta = \alpha^{1/q}, \ \beta_1 = \beta\alpha^{-1}(1/q), \ (i+1)\beta_{i+1} = \beta_i\alpha^{-1}(1/q - i) \ \ (i = 1, \cdots, h-2),$$

and thus we have found the branches needed in the construction. Since $\alpha^{1/q}$ has q values unless $\alpha = 0$, we have found the q^r (or q^{r-1} if some α_i is 0) solutions W which are polynomials in A.

PURDUE UNIVERSITY.

ON SOME CUBIC DIOPHANTINE EQUATIONS.*

By E. ROSENTHALL.

1. In this paper we shall adhere to the following notation: large capital letters $A, B, \cdots$ (with or without subscripts) will represent integers of the quadratic number field $Ra(\rho)$ where $\rho = \frac{1}{2}(-1 + i\sqrt{3})$; the letter ϵ will be reserved for the units of this field. Small latin letters $a, b, \cdots$ represent rational integers, and the conjugate of a number X is denoted by $\bar{X}$. The symbol $(A, B, C, \cdots)$ signifies the g. c. d. of $A, B, C, \cdots$; $A \mid B$ means A divides B, and $(A, B) = 1$ denotes that A and B are coprime.

The integers of the field $Ra(\rho)$ are of the form $c + d\rho$, or of the form $\frac{1}{2}(a + \sqrt{-3b})$ with a and b of the same parity; also $\epsilon\bar{\epsilon} = 1$.

2. In this paper we solve completely in integers of $Ra(\rho)$ a multiplicative equation of the form

$$wW\bar{W} = nN\bar{N} \tag{2.1}$$

and deduce from this the complete rational integer solution of certain interesting cubic diophantine equations. The method of proof will indicate how some types of diophantine equations reducible in a quadratic field may be solved completely.

Since the integers of $Ra(\rho)$ obey the fundamental law of arithmetic, namely unique decomposition into prime factors, multiplicative equations in this field can be solved completely in parametric form by the method of reciprocal arrays.[1]

To solve (2. 1) we require the following fundamental lemma.

LEMMA 1.[1] *All integral solutions of*

$$XY = ZW$$

are given by $X = US$, $Y = VT$, $Z = UT$, $W = VS$; *and it suffices to take* $(S, T) = 1$.

This lemma (as stated in the paper referred to) is for non-zero integers X, Y, Z, W. However the solution given includes all the possible zero solu-

* Received August 19, 1942; Revised February 19, 1943.

[1] E. T. Bell, "Polynomial diophantine systems," *Transactions of the American Mathematical Society*, vol. 35 (1933), pp. 909-910. Also E. T. Bell, "Reciprocal arrays and diophantine analysis," *American Journal of Mathematics*, vol. 35 (1933), pp. 50-66.

tions without violating the conditions on S and T. For example, the solution $X = Z = W = 0$ is obtained by taking $U = S = 0$, $T = 1$, and S, T are coprime.

3. We now solve equation (2.1).

THEOREM 1. *The complete solution in integers w, W, n, N satisfying equation (2.1) is given by*

$$W = SUL, \quad w = tV\bar{V}, \quad N = S\bar{U}\bar{V}, \quad n = t\bar{L}\bar{L} \tag{3.1}$$

and it suffices to take $(UL, \bar{U}\bar{V}) = (V\bar{V}, L\bar{L}) = 1$.

Proof. By Lemma 1 all required values of w, W, n, N are of the form

$$\begin{aligned} W &= RAF & N &= RBH \\ \bar{W} &= SBG & \bar{N} &= SCF \\ w &= TCH & n &= TAG \end{aligned} \tag{3.2}$$

with the g. c. d. conditions

$$(W, N) = R, \quad (\bar{W}, \bar{N}) = S, \quad (w, n) = T. \tag{3.3}$$

From (3.3) it follows that $R = \epsilon\bar{S}$; substituting this in (3.2) we see that the parameters must be restricted to satisfy

$$\epsilon AF = \bar{B}\bar{G} \quad \text{and} \quad \epsilon BH = \bar{C}\bar{F}. \tag{3.4}$$

Hence (3.2) becomes

$$\begin{aligned} W &= \bar{S}\bar{B}\bar{G} & N &= \bar{S}\bar{C}\bar{F} \\ w &= TCH & n &= TAG \end{aligned} \tag{3.5}$$

with conditions (3.4). By Lemma 1 all A, F, B, G satisfying $(3.4)_1$ are given by

$$\bar{B} = A_1B_1, \quad \bar{G} = C_1D_1, \quad \epsilon A = A_1D_1, \quad F = C_1B_1.$$

Then $(3.4)_2$ becomes $\epsilon\bar{A}_1H = \bar{C}\bar{C}_1$ and so

$$\epsilon\bar{A}_1 = A_2B_2, \quad H = C_2D_2, \quad \bar{C} = A_2D_2, \quad \bar{C}_1 = C_2B_2.$$

Substituting these values in (3.5) we have

$$\begin{aligned} W &= \epsilon\bar{S}\bar{A}_2\bar{C}_2(\bar{B}_2B_1)(\bar{B}_2D_1), & N &= \bar{S}A_2C_2D_2(B_2\bar{B}_1) \\ w &= T\bar{A}_2C_2(D_2\bar{D}_2), & n &= T\bar{A}_2C_2(\bar{B}_2D_1)(B_2\bar{D}_1). \end{aligned}$$

Since the integers B_2, B_1, D_1 always occur as factors of the products $\bar{B}_2B_1, \bar{B}_2D_1$ we can put $\bar{B}_2B_1 = K$ and $\bar{B}_2D_1 = L$; we also make the reversible substitution $D_2 = \epsilon\bar{V}$ and replace $\epsilon\bar{S}$ by the parameter S. Then we have

$$(3.6)\quad W = S(\bar{A}_2\bar{C}_2K)L,\ w = (T\bar{A}_2C_2)V\bar{V},\ N = S(A_2C_2\bar{K})\bar{V},\ n = (T\bar{A}_2C_2)L\bar{L},$$

where $T\bar{A}_2C_2$ must be restricted to be a rational integer. Thus (3.6) is a complete parametric representation for the indeterminates satisfying (2.1).

However if we put $\bar{A}_2\bar{C}_2K = U$, $T\bar{A}_2C_2 = t$ then all numbers in (3.6) are included in (3.1). But by direct substitution we see that all the numbers (3.1) are solutions of (2.1). Hence (3.1) is the complete solution of (2.1) where S, U, L, V are arbitrary integers in $Ra(\rho)$, and t is any rational integer.

4. From Theorem 1 we can deduce the complete rational integer solution of $x^3 + y^3 = z^3 + w^3$. This equation can be put in the form (2.1), viz.,

$$(4.1)\qquad (X + \bar{X})X\bar{X} = (Y + \bar{Y})Y\bar{Y}$$

where $X = \frac{1}{2}[(x+y) + \sqrt{-3}(x-y)]$, $Y = \frac{1}{2}[(z+w) + \sqrt{-3}(z-w)]$ are integers of $Ra(\rho)$. If $x = y$, we have the trivial solutions with $x = z$.

In **6** we prove the following theorem.

THEOREM 2. *The set of all integers X, Y (with $X \neq Y$) satisfying (4.1) is given by*

$$(4.2)\qquad X = (\sqrt{-3}p/\theta q)(e + f\rho)LU,\quad Y = (\sqrt{-3}p/\theta q)(e + f\rho)\bar{U}\bar{V}$$

where $e + f\rho = UV \cdot V\bar{V} - \bar{U}\bar{L} \cdot \bar{L}L$, and θ is 1 unless $\sqrt{-3} \mid (e + f\rho)$ and then $\theta = 3$; it suffices to take $q = (e, f)$ and $(LU, \bar{U}\bar{V}) = (V\bar{V}, L\bar{L}) = 1$.

5. To obtain the solution of (4.1) in the form (4.2) we require the following three lemmas.

LEMMA 2. *If the product XY is pure imaginary and if $X = aA$ where A is not divisible by a rational integer then $Y = \sqrt{-3}b\bar{A}/\theta$ where θ is 1, or 3 if $\sqrt{-3} \mid A$.*

Proof. Put $X = aA = a(m + n\rho)$ and $Y = b(s + t\rho)$ where $(m, n) = (s, t) = 1$. Then since XY is pure imaginary we must have

$$ns + mt - nt = 2(ms - nt)$$

from which it follows that $m(2s - t) = n(s + t)$; and since $(m, n) = 1$ we have $2s - t = kn$, $s + t = km$. Hence $s = k(n + m)/3$, $t = k(2m - n)/3$ and since $(s, t) = 1$ we can write

$$s = (n+m)/\theta, \qquad t = (2m-n)/\theta,$$

which gives $Y = b\{(m+n) + (2m-n)\rho\}/\theta = b\sqrt{-3A}/\theta$.

LEMMA 3. *If $D = (l + m\rho)$ is a g.c.d. of $a + b\rho$ and $c + d\rho$, then $(a, b, c, d) = (l, m)$.*

Proof. We know that if a rational integer t divides $a + b\rho$ then t divides both a and b.

Therefore if $q = (l, m)$, then $q \mid D$ and q is a divisor of a and b, c and d; and if $s = (a, b, c, d)$ then $q \mid s$. Also s divides $a + b\rho$, $c + d\rho$, and therefore $s \mid D$. Then $s \mid l$ and $s \mid m$, and therefore $s \mid q$. Hence $s \mid q$, $q \mid s$ and so $s = q$, $(a, b, c, d) = (l, m)$.

LEMMA 4. *If the matrix of the coefficients is of rank 2, then all integral solutions of the simultaneous equations*

$$AX + BY + CZ = 0, \qquad A_1X + B_1Y + C_1Z = 0$$

are given by

$$X = (E/D)(BC_1 - B_1C), Y = (E/D)(A_1C - AC_1), Z = (E/D)(AB_1 - A_1B)$$

where E is an arbitrary integer and it suffices to take $D = (BC_1 - B_1C, A_1C - AC_1, AB_1 - A_1B)$.

For, solving algebraically, X, Y, Z are certainly given by

$$X = \alpha(BC_1 - B_1C), \quad Y = \alpha(A_1C - AC_1), \quad Z = \alpha(AB_1 - A_1B)$$

where α is in the field; writing α in lowest terms E/D, D must be a divisor of $BC_1 - B_1C$, etc. Hence, multiplying up by a suitable factor, it suffices to take D as stated, and E is arbitrary.

The case where the matrix is of rank 0 or 1 will be considered in the application to which the lemma is put.

6. We now prove Theorem 2. Considering each element of (4.1) as independent, then by Theorem 1 all X, Y satisfying (4.1) are of the form

$$(6.1) \qquad X = SUL, \quad X + \bar{X} = tV\bar{V}, \quad Y = S\bar{U}\bar{V}, \quad Y + \bar{Y} = tL\bar{L}$$

with $(UL, \bar{U}\bar{V}) = (V\bar{V}, L\bar{L}) = 1$ and the parameters must be restricted to satisfy

$$SUL + \bar{S}\bar{U}\bar{L} - tV\bar{V} = 0, \qquad S\bar{U}\bar{V} + \bar{S}UV - tL\bar{L} = 0.$$

Considering this system in the indeterminates S, $\bar{S}$, t and taking the matrix of the coefficients to be of rank 2, then by Lemma 4 all S, $\bar{S}$ are given by

$$S = (E/D)(e + f\rho), \qquad \bar{S} = -(E/D)(e + f\rho^2)$$

where E is arbitrary and it suffices to take $D = (e + f\rho, e + f\rho^2)$. From the expressions for S and $\bar{S}$ we get $E\bar{D} + \bar{E}D = 0$, and hence $E\bar{D}$ is pure imaginary; thus by Lemma 2 we can put

$$D = (k + l\rho) = qK, \qquad E = \sqrt{-3}\, pK/\theta \text{ with } q = (k, l),$$

and $\theta = 1$ unless $\sqrt{-3} \mid K$, that is $\sqrt{-3} \mid (e + f\rho)$ and then $\theta = 3$; from Lemma 3 it follows that $q = (e, f)$. Hence

$$E/D = \sqrt{-3}\, p/\theta q$$

where $q = (e, f)$, and Theorem 2 is proved for the case considered.

If the matrix of the coefficients is of rank 1 then $\bar{\epsilon}V = \epsilon L = \bar{U}$ or $U = 0$ giving in (6.1) the trivial solution $X = Y$; and if the matrix is of rank 0 we get $X = Y = 0$, a solution which is included in (4.2).

7. From Theorem 2 we deduce the following result.

THEOREM 3. *All sets of rational integral values x, y, z, w (except $x = z$) satisfying the equation*

$$x^3 + y^3 = z^3 + w^3 \tag{7.1}$$

are given by

$$x = p(a - 2b)/\theta q,\ y = -p(a + b)/\theta q,\ z = p(c - 2d)/\theta q,\ w = -p(c + d)/\theta q$$

where $a + b\rho = UL(e + f\rho)$, $c + d\rho = \bar{U}\bar{V}(e + f\rho)$, and q, θ, $e + f\rho$, are as defined in Theorem 2.

For from Theorem 2 we have

$$\tfrac{1}{2}[(x + y) + \sqrt{-3}(x - y)] = (\sqrt{-3}p/\theta q)(a + b\rho),$$
$$\tfrac{1}{2}[(z + w) + \sqrt{-3}(z - w)] = (\sqrt{-3}p/\theta q)(c + d\rho).$$

Equating real and imaginary parts and solving for x, y, z, w yields the required results. Further it suffices to take $(UL, \bar{U}\bar{V}) = (V\bar{V}, L\bar{L}) = 1$.

8. The complete solution of the equation $mM\bar{M} = nN\bar{N}$ also yields the complete rational integral solution [2] of

$$x^3 + y^3 + z^3 - 3xyz = u^3 + v^3 + w^3 - 3uvw, \tag{8.1}$$

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[2] R. D. Carmichael, *Diophantine Analysis*, New York, 1915, pp. 63-65, discusses this equation and gives a four parameter rational solution.

where

$$m = x + y + z, \quad M = (x - z) + \rho(y - z), \quad n = u + v + w,$$
$$N = (u - w) + \rho(v - w).$$

Then from (3.1) all rational integers satisfying (8.1) are obtainable from

$$(8.2)\quad \begin{aligned} &x + y + z = tV\bar{V} \equiv a, && u + v + w = tL\bar{L} \equiv a_1, \\ &(x - z) + \rho(y - z) = SUL \equiv b + c\rho, && (u - w) + \rho(v - w) = S\bar{U}\bar{V} \equiv b_1 + \end{aligned}$$

and it suffices to take $(L\bar{L}, V\bar{V}) = (UL, \bar{U}\bar{V}) = 1$. Solving (8.2) for x, y, z, u, v, w gives

$$(8.3)\quad \begin{aligned} &x = \tfrac{1}{3}(a + 2b - c), && y = \tfrac{1}{3}(a - b + 2c), && z = \tfrac{1}{3}(a - b - c) \\ &u = \tfrac{1}{3}(a_1 + 2b_1 - c_1), && v = \tfrac{1}{3}(a_1 - b_1 + 2c_1), && w = \tfrac{1}{3}(a_1 - b_1 - c_1) \end{aligned}$$

for the complete rational integral solution of (8.1).

It is desirable to know how the parameters t, U, L, V are to be selected so that (8.3) always gives integers. These expressions are integers if $a + 2b - c \equiv 0 \pmod 3$ and $a_1 + 2b_1 - c_1 \equiv 0 \pmod 3$. This implies

$$(8.4)\quad \begin{aligned} &tV\bar{V} + SUL + \bar{S}\bar{U}\bar{L} \equiv 0 \pmod 3 \\ &tL\bar{L} + S\bar{U}\bar{V} + \bar{S}UV \equiv 0 \pmod 3. \end{aligned}$$

If both UL and $\bar{U}\bar{V}$ are prime to 3, then it suffices to take S satisfying the linear congruence $AS \equiv B \pmod 3$, where $UL \cdot UV - \bar{U}\bar{L} \cdot \bar{U}\bar{V} = A$, and $-t(V\bar{V}\,\bar{U}\bar{V} - L\bar{L} \cdot UL) = B$. This congruence is solvable if $(3, A) \mid B$. Since A is divisible by $\sqrt{-3}$, B will have to contain the factor $\sqrt{-3}$ or $(\sqrt{-3})^2$. If it happens that U, L, V are selected so that $(V\bar{V} \cdot UV - UL \cdot L\bar{L})$ does not contain this factor then it suffices to select $t \equiv 0 \pmod 3$; otherwise t is arbitrary.

If one of $UL, \bar{U}\bar{V}$ (say UL) is not prime to 3, then it suffices to take $\bar{U}\bar{V}$ prime to 3; whence U must be prime to 3. Thus L must be divisible by $\sqrt{-3}$, and it follows therefore that $L\bar{L} \equiv 0 \pmod 3$ and $SUL + \bar{S}\bar{U}\bar{L} \equiv 0 \pmod 3$. Hence (8.4) becomes

$$(8.5)\qquad tV\bar{V} \equiv 0 \pmod 3, \qquad S\bar{U}\bar{V} + \bar{S}UV \equiv 0 \pmod 3.$$

From $(8.5)_1$ we have $t \equiv 0 \pmod 3$ since $V\bar{V}, L\bar{L}$ are coprime; and $(8.5)_2$ holds if and only if $\sqrt{-3}$ divides $S\bar{U}\bar{V}$, whence $\sqrt{-3}$ divides S since $\bar{U}\bar{V}$ is prime to 3. Summing up, we have, in this case, that (8.3) yields integers if and only if S, L (or V) is divisible by $\sqrt{-3}$, and $t \equiv 0 \pmod 3$.

9. As a further illustration of the method used here for solving diophantine equations, we now solve completely in rational integers the equation [3]

$$x^3 + y^3 = z^2. \tag{9.1}$$

For this purpose we require the following lemmas.

LEMMA 5. *If the product XYZ is a rational integer then all values of X, Y, Z are given by*

$$X = aAB, \quad Y = b\bar{A}C, \quad Z = c\bar{B}\bar{C}.$$

Proof. Precisely as in Lemma 2 it follows that if the product RS is a rational integer then all R, S are given by $R = aK$, $S = p\bar{K}$. Hence since $X(YZ)$ is rational we can put $X = aK$, $YZ = p\bar{K}$; whence by Lemma 1, we have

$$Y = FG, \quad Z = HJ, \quad p = FJ, \quad \bar{K} = HG.$$

Again, since FJ is rational it suffices to take $F = bC$, $J = c\bar{C}$; make the change in parameters $H \to \bar{B}$, $G \to \bar{A}$, and the lemma is proved.

We note that it suffices to take A, cC coprime (Lemma 1); further any factor common to a and c can be absorbed into the B and so we can take $(a, c) = 1$. These facts will be required in **10**.

LEMMA 6. *The complete set of integer values X, Y satisfying $X\bar{X} = Y\bar{Y}$ is given by $X = \epsilon ST$, $Y = S\bar{T}$, and it suffices to take T divisible by no rational integer.*

Proof. This equation $X\bar{X} = Y\bar{Y}$ can be considered as a particular case of Theorem 1 by placing $w = n = 1$, from which it follows in (3.1) that $t = 1$, $V = \epsilon_1$, $L = \epsilon_2$. On making the reversible substitution $U = \bar{\epsilon}_1 T$, $\bar{\epsilon}_1\epsilon_2 = \epsilon$ we obtain the required result.

LEMMA 7. *All integers X, z satisfying $X\bar{X} = z^2$ are given by $X = \epsilon aT^2$, $z = aT\bar{T}$.*

For, from Lemma 6 all required X, z are given by $X = \epsilon ST$, $z = S\bar{T}$, where $S\bar{T}$ must be a rational integer; then we can put $S = aT$, since T is divisible by no rational integer.

Lemmas 6 and 7 furnish the complete rational integral solutions of each of the equations [4] $x^2 + my^2 = z^2 + mw^2$ and $x^2 + my^2 = u^2$ for those m in

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[3] L. E. Dickson, *History of the Theory of Numbers*, vol. 2, pp. 578-581 gives an account of the investigations on (9.1).

[4] If $m \equiv 1\,(4)$ replace form $u^2 + mv^2$ by $u^2 + uv + \frac{1}{4}(1 - m)v^2$.

which the integers of the quadratic field $Ra(\sqrt{-m})$ are uniquely decomposable into prime factors. The complete solutions of these equations for any value of m are readily obtained by other methods,[5] but for our purpose the form of solution given here is desirable since it occurs in multiplicative form.

10. We now return to equation (9. 1). Defining the integer X as in **4**, this equation takes the form

$$(X+\bar{X})X\bar{X} = z\cdot z\cdot 1. \tag{10. 1}$$

Then from (3. 1) it follows that all X, $X+\bar{X}$, z of (10. 1) are given by

$$X = SUL, \quad X+\bar{X} = tV\bar{V}, \quad z = S\bar{U}\bar{V}, \tag{10. 2}$$

where t and L must satisfy $tL\bar{L}=1$. Thus $t=1$ (since $L\bar{L}>0$), $L=\epsilon$; and with the reversible substitution $U=\bar{\epsilon}K$, $V=\epsilon P$, (10. 2) becomes

$$X = SK, \quad X+\bar{X} = P\bar{P}, \quad z = S\bar{K}\bar{P},$$

and it suffices to take K divisible by no rational integer (it can be absorbed into S).

Since $S\bar{K}\bar{P}$ must be a rational integer, then

$$S = aAB, \quad \bar{K} = \bar{A}C, \quad \bar{P} = c\bar{B}\bar{C}$$

and it suffices here to have c prime to a and A (see remarks at the end of Lemma 5), and we have

$$X = aA^2R, \quad X+\bar{X} = c^2R\bar{R}, \quad z = acA\bar{A}R\bar{R} \tag{10. 3}$$

in which, since B and C always occur as the product $B\bar{C}$, we have put $B\bar{C}=R$.

Further, the parameters of (10. 3) must satisfy $c^2R\bar{R} - aA^2R - a\bar{A}^2\bar{R} = 0$ which is equivalent to

$$(c^2R - a\bar{A}^2)(c^2\bar{R} - aA^2) = (aA\bar{A})^2.$$

Put $c^2R - a\bar{A}^2 = H$, then (10. 3) becomes

$$X = aA^2\left(\frac{H+a\bar{A}^2}{c^2}\right), \qquad z = acA\bar{A}\left(\frac{H+a\bar{A}^2}{c^2}\right)\left(\frac{\bar{H}+aA^2}{c^2}\right) \tag{10. 4}$$

where $(aA\bar{A})^2 = H\bar{H}$. Whence, by Lemma 1,

$$H = \epsilon bT^2, \quad aA\bar{A} = bT\bar{T}. \tag{10. 5}$$

Solving $(10. 5)_2$ by Theorem 1 we finally get

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[5] L. E. Dickson, *Introduction to the Theory of Numbers*, pp. 40-41; also see Ex. 5, p. 43.

$$(10.6) \qquad H = \epsilon m G\bar{G}(M\bar{F}\bar{E})^2, \quad a = mE\bar{E}, \quad A = MFG.$$

Again, since R is an integer we must have c^2 dividing $H + a\bar{A}^2$, that is it must divide $m\bar{G}\bar{F}^2\bar{E}(\epsilon GM^2\bar{E} + E\bar{M}^2\bar{G})$. Since c is prime to a and A, then we have c^2 prime to each of m, E, F, G, M. Hence c^2 must divide $(E\bar{M}^2\bar{G} + \epsilon\bar{E}M^2G)$.

Substituting (10.6) in (10.4) we have

$$X = (mF\bar{F})^2(E\bar{G})(\bar{E}G)^2M^2\left(\frac{E\bar{M}^2\bar{G} + \epsilon\bar{E}M^2G}{c^2}\right),$$

$$z = c(mF\bar{F})^3(\bar{E}G)^2(E\bar{G})^2M\bar{M}\left(\frac{E\bar{M}^2\bar{G} + \epsilon\bar{E}M^2G}{c^2}\right)\left(\frac{\bar{E}M^2G + \bar{\epsilon}E\bar{M}^2\bar{G}}{c^2}\right).$$

Put $mF\bar{F} = k$ and $E\bar{G} = K$, and we get

$$(10.7) \qquad \begin{aligned} X &= k^2K^2\bar{K}M^2\left(\frac{\bar{K}\bar{M}^2 + \epsilon KM^2}{c^2}\right) \\ z &= \bar{\epsilon}ck^3(K\bar{K})^2M\bar{M}\left(\frac{\bar{K}\bar{M}^2 + \epsilon KM^2}{c^2}\right), \end{aligned}$$

and it suffices to have c prime to each of K, M and k.

All the integer solutions of (10.1), and hence of (9.1), are given by (10.7), and in order to obtain all the integers in (10.7) we can take c, k arbitrary and M any integer prime to c and then select the coördinates of K to make $\left(\frac{\bar{K}\bar{M}^2 + \epsilon KM^2}{c^2}\right)$ integral. This is done in the following way.

Put $K = u + v\rho$, and let $M^2 = m + n\rho$, (whence m, n, c^2 are coprime). Then if we take $\epsilon = 1, \rho, \rho^2$ in succession the congruence $\bar{K}\bar{M}^2 + \epsilon KM^2 \equiv 0 \pmod{c^2}$ becomes

$$u(2m - n) \equiv (m + n)v \pmod{c^2}, \quad u(m - 2n) \equiv (2m - n)v \pmod{c^2},$$
$$u(m + n) \equiv (2n - m)v \pmod{c^2}$$

respectively; and for $\epsilon = -1, -\rho, -\rho^2$ respectively we get the congruences

$$un \equiv v(n - m) \pmod{c^2}, \quad um \equiv vn \pmod{c^2}, \quad u(n - m) \equiv vm \pmod{c^2}.$$

Since c^2 is prime to at least one of the coördinates m or n it follows that in the last three congruences either the coefficient of u or of v (at least) must be prime to c^2. If this coefficient is the coefficient of u, then select v arbitrary and solve the congruence for the unique u, and similarly solve for the unique v with arbitrary u if the coefficient of v is prime to c^2.

In the first three congruences, also one of the coefficients (that is, of u or v) must be prime to c^2, and so we can always find a unique u or v. For

example, (considering the first congruence), suppose that $2m - n$, $m + n$, c^2 have the factor $p \neq 3$ in common. Then $p \mid 3m$ and hence divides m, n, c^2, which is impossible since c^2, M^2 are coprime.

Also if 3 is a factor of c^2 then $2m - n$ (and hence $m + n$) is prime to 3, for otherwise $M^2 = \frac{1}{2}(2m - n + \sqrt{-3n})$ would be divisible by $\sqrt{-3}$, and so M^2 would have a factor in common with c^2.

Thus in all cases either the coefficient of u or v is prime to c^2, and K can be found.

11. The equations which we have solved completely in rational integers, viz. (7.1), (8.1), and (9.1) are merely particular cases of the equation (2.1). By suitably selecting the indeterminates w, W, n, N in (2.1) the complete solution to many other interesting equations can be obtained; for example any of the equations $x^3 + y^3 + z^3 - 3xyz = u^2$, $u^2 + 3v^2$, $u^3 + v^3$; or $(x^3 + y^3)(u^2 + v^2) = (w^2 + z^3)$. Moreover Theorem 1 holds in any normal algebraic field whose integers can be decomposed uniquely into prime factors.

McGILL UNIVERSITY,
MONTREAL, CANADA.

A NOTE ON SOUSLIN'S PROBLEM.*

By Edwin W. Miller.

As is well known, the following properties characterize the linear continuum L as a type of linear order.

(1) L has no first and no last element.

(2) L is continuous; i. e., every cut in L is a Dedekind cut.

(3) There exists a denumerable subset D of L such that between any two elements of L there is an element of D.

As an immediate consequence of (3) we have

(4) Any set of non-overlapping intervals on L is countable.

Souslin raised the following question.[1] Do properties (1), (2) and (4) characterize the linear continuum? In other words, do properties (1), (2) and (4) imply property (3)?

The purpose of the present article is to prove the following

Theorem. *In order that there exist a linear order which possesses properties* (1), (2) *and* (4) *without possessing property* (3) *it is necessary and sufficient that there exist a partial order P of power $\aleph_1$ such that*

(a) *if $Q \subset P$ and $\bar{\bar{Q}} = \aleph_1$, then Q contains two comparable elements and two non-comparable elements;*

(b) *if x and y are non-comparable elements of P, then there exists no z in P such that $x < z$ and $y < z$.*

Proof of the necessity. Suppose there exists a linear order M possessing properties (1), (2) and (4) but not property (3). Let I_1 be any interval on M, and let α be any ordinal $< \Omega$. Suppose an interval I_β has been defined on M for every $\beta < \alpha$. There must exist an interval I on M which contains no endpoint of any interval I_β $(\beta < \alpha)$, for otherwise the denumerable set consisting of the end points of the intervals I_β $(\beta < \alpha)$ would be dense in M, and (3) would be satisfied. Let I_α be any proper subinterval of such an interval I. Clearly, if $\beta < \alpha$, then either I_α is a proper subinterval of I_β or else $I_\alpha \cdot I_\beta = 0$.

* Received July 10, 1942. Author died July 23, 1942.

[1] *Fundamenta Mathematicae*, vol. 1 (1920), p. 223.

We now define a partial order P whose elements are the intervals I_α $(\alpha < \Omega)$ by setting $I_\beta < I_\alpha$ if and only if $I_\beta \supset I_\alpha$. Clearly, $\bar{\bar{P}} = \aleph_1$. Furthermore, if I_α and I_β are non-comparable then $I_\alpha \cdot I_\beta = 0$, and (b) is at once seen to be satisfied. To see that (a) is satisfied, let Q be any nondenumerable subset of P. Now Q must contain two comparable elements, for otherwise there is determined a nondenumerable infinity of non-overlapping intervals on M. Suppose that every two elements of Q were comparable. It follows that there would exist a decreasing Ω-sequence of intervals on M: $I^{(1)}, I^{(2)}, \cdots I^{(\alpha)}, \cdots (\alpha < \Omega)$. Let a_α be the left-hand endpoint of $I^{(\alpha)}$. The intervals $(a_\alpha, a_{\alpha+1})$ do not overlap, and a contradiction of (4) is obtained.

In proving the conditions sufficient we shall make use of the following theorems.[2]

THEOREM A. *If P is a partial order of power $\aleph_1$, and every subset of P of power $\aleph_1$ contains two comparable elements, then almost every*[3] *element of P is comparable with $\aleph_1$ elements of P.*

THEOREM B. *If P is a partial order of power $\aleph_1$, and every subset of P of power $\aleph_0$ contains two comparable elements, then P contains a linear order of power $\aleph_1$.*

Let B be a partial order, and A a subset of B such that the elements of A are mutually incomparable. We shall say that A is *maximal* (relative to B) provided $x \in B - A$ implies that x is comparable with some element of A.

The notation $x \phi y$ will be used to mean that x and y are incomparable.

Proof of the sufficiency. Let P be a partial order of power $\aleph_1$ possessing properties (a) and (b). In virtue of Theorem A, almost every element of P is comparable with $\aleph_1$ elements of P. Furthermore, it follows from (a) and (b) that for every element x of P, at most $\aleph_0$ elements of P precede x. Accordingly we can remove a countable set from P so as to obtain a partial order P' (of power $\aleph_1$) which possesses the following properties.

(a′) If $Q \subset P'$ and $\bar{Q} = \aleph_1$, then Q contains two comparable elements and two non-comparable elements.

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[2] Theorem A is an immediate consequence of Lemma 5.21 in a paper by Ben Dushnik and the author, entitled "Partially ordered sets," *American Journal of Mathematics*, vol. 63 (1941), pp. 600-610. Theorem B is a special case of Theorem 5.23 of the same paper.

[3] By "almost every" element of P is meant every element with the exception of a countable set.

(b′) $x, y \in P'$, $x \phi y$ implies that there exists no z in P' such that $x < z$ and $y < z$.

(c′) $x \in P'$ implies that there exist $\aleph_1$ elements of P' which follow x.

(d′) $x \in P'$ implies that at most $\aleph_0$ elements of P' precede x.

Now let A_1 be a maximal set of mutually incomparable elements of P' such that $\bar{\bar{A}}_1 = \aleph_0$. The existence of A_1 follows from (a′) and Theorem B. Let Q_1 denote the set of all x in P' such that x precedes an element of A_1. In virtue of (d′) we have $\bar{\bar{Q}}_1 \leqq \aleph_0$. Let α be any ordinal $< \Omega$. Suppose that A_β and Q_β have been defined for all $\beta < \alpha$ so that $\bar{\bar{A}}_\beta = \aleph_0$, $\bar{\bar{Q}}_\beta \leqq \aleph_0$, the elements of A_β are mutually incomparable, $A_\beta \subset P' - \sum_{\mu<\beta} (A_\mu + Q_\mu)$ and is maximal relative to $P' - \sum_{\mu<\beta} (A_\mu + Q_\mu)$, and Q_β is the set of all x in P' such that $x <$ an element of A_β. We shall now define A_α and Q_α.

Case 1. α is a limiting ordinal.

In this case, we merely take A_α so that $\bar{\bar{A}}_\alpha = \aleph_0$, the elements of A_α are mutually incomparable, $A_\alpha \subset P' - \sum_{\beta<\alpha} (A_\beta + Q_\beta)$ and is maximal relative to $P' - \sum_{\beta<\alpha} (A_\beta + Q_\beta)$. We then define Q_α as the set of all x in P' such that $x <$ an element of A_α. We have $\bar{\bar{Q}}_\alpha \leqq \aleph_0$.

Case 2. α is not a limiting ordinal.

Denote the immediate predecessor of α by γ, and for any x in A_γ denote by B_x the set of all elements of $P' - \sum_{\beta<\alpha} (A_\beta + Q_\beta)$ which follow x. We have $\bar{\bar{B}}_x = \aleph_1$. Let C_x be such that $\bar{\bar{C}}_x = \aleph_0$, the elements of C_x are mutually incomparable, $C_x \subset B_x$ and is maximal relative to B_x. We now denote $\sum_{x \in A_\gamma} C_x$ by A_α. We have $\bar{\bar{A}}_\alpha = \aleph_0$. Let Q_α denote the set of all x in P' such that $x <$ an element of A_α. We have $\bar{\bar{Q}}_\alpha \leqq \aleph_0$. Now $A_\alpha \subset P' - \sum_{\beta<\alpha} (A_\beta + Q_\beta)$. We shall show that the elements of A_α are mutually incomparable and that A_α is maximal relative to $P' - \sum_{\beta<\alpha} (A_\beta + Q_\beta)$.

Let a and b be any two elements of A_α. If there exists an x in A_γ such that $a + b \subset C_x$, then certainly $a \phi b$. Suppose $a \in C_x$, $b \in C_y$, $x \neq y$. Assume $a < b$. Then $x < a$, since $a \in C_x$ and $C_x \subset B_x$. Hence $x < b$. But $y < b$ since $b \in C_y$ and $C_y \subset B_y$. However, $x \phi y$ since the elements of A_γ are mutually incomparable. But this contradicts (b′).

Now suppose that A is not maximal relative to $P' - \sum_{\beta<\alpha} (A_\beta + Q_\beta)$. Then there exists an x in $P' - \{\sum_{\beta<\alpha} (A_\beta + Q_\beta) + A_\alpha\}$ such that $x \phi A_\alpha$. Since

$x \epsilon P' - \sum_{\beta<\alpha} (A_\beta + Q_\beta)$ and A_γ is maximal relative to $P' - \sum_{\beta<\gamma} (A_\beta + Q_\beta)$, there exists an element y of A_γ such that $y < x$. Hence $x \epsilon B_y$ and since $x \phi A_\alpha$, we have $x \phi C_y$. But this contradicts the fact that C_y is maximal relative to B_y.

We now put $A = \sum_{\alpha<\Omega} A_\alpha$.

We shall make use of the following properties of the partial order A.

(I) $A = \sum_{\alpha<\Omega} A_\alpha$; $\bar{\bar{A}} = \aleph_1$, $\bar{\bar{A}}_\alpha = \aleph_0$, $A_\alpha \cdot A_\beta = 0$ if $\alpha \neq \beta$.

(II) The elements of A_α are mutually incomparable.

(III) $x \epsilon A_\alpha$ implies that $x < \aleph_0$ elements of $A_{\alpha+1}$.

(IV) $\beta < \alpha$, $x \epsilon A_\beta$, $y \epsilon A_\alpha$ implies that $x < y$ or $x \phi y$.

(V) $\beta < \alpha$, $x \epsilon A_\alpha$ implies that there exists one and only one element x_β of A_β such that $x_\beta < x$; furthermore we have $x_1 < x_2 \cdots x_\beta \cdots < x$.

(VI) $Q \subset A$, $\bar{\bar{Q}} = \aleph_1$ implies that Q contains two comparable elements and two non-comparable elements.

Properties I, II, III, IV and VI are obvious from the way in which A was obtained. To prove V we proceed as follows. Let $x \epsilon A_\alpha$, and let β be any ordinal $< \alpha$. There exists an element x_β of A_β such that x_β is comparable with x, since A_β is maximal relative to $P' - \sum_{\mu<\beta} (A_\mu + Q_\mu)$. In virtue of IV we have $x_\beta < x$. In virtue of II and (b) there cannot be two such elements x_β. Now let μ and ν be ordinals such that $\mu < \nu < \alpha$. We cannot have $x_\mu \phi x_\nu$ since $x_\mu < x$ and $x_\nu < x$. In virtue of IV we must have $x_\mu < x_\nu$. Hence $x_1 < x_2 < \cdots x_\beta < \cdots < x$.

We now proceed to define a linear extension L of A which has the following properties.

(*) L has no first element and no last element.

(**) No denumerable set is dense in L.

(***) Any set of non-overlapping intervals on L is countable.

First, we write the elements of A_1 in a series of type $\omega^* + \omega$: $\cdots a_{-n}, \cdots a_{-1}, a_0, a_1, \cdots a_n, \cdots$.

Next, we write the elements of A_2 in the form a_{nm}, where m runs through all integers to give the set A_2, where the n indicates that $a_n < a_{nm}$, and where, for a fixed n, the integers m form a series of type $\omega^* + \omega$.

Referring to property V, and using the obvious induction, we may assume that the elements of A_α, for any $\alpha < \Omega$, have been given the form $a_{n_1 n_2 \ldots n_\mu \ldots n_\alpha}$, where n_α runs through all the integers to give the whole set A_α, and $a_{n_1 n_2 \ldots n_\mu} < a_{n_1 n_2 \ldots n_\mu \ldots n_\alpha}$. Furthermore, in case α is of the form

$\gamma+1$ and $a_{n_1 n_2 \dots n_\gamma}$ is a (fixed) element of A_γ, we may assume that the integers n_α appearing in the terms $a_{n_1 n_2 \dots n_\gamma n_\alpha}$ of A_α form a series of type $\omega^* + \omega$.

Now, if x' and x'' are any two elements of A, we shall put $x' < x''$ in L if and only if (1) the series subscript of x' is a lower segment of the series subscript of x'', or (2) $n_\alpha' < n_\alpha''$, where α indicates the first place where the series subscripts of x' and x'' differ.

It is easily verified that L is a linear extension of A.

Let x be any element of L. Let n_1 denote the first term in the series subscript of x. Let n'_1 be an integer $< n_1$. Let $x' = a_{n'_1}$. It is clear that $x' < x$, so that L has no first element. In a similar way we prove that L has no last element. Thus (*) is verified.

To prove (**), let D be any denumerable subset of L. There exists an $\alpha < \Omega$ such that $D \subset \sum_{\beta<\alpha} A_\beta$. Let x be an element of A_α and y an element of $A_{\alpha+1}$ such that $x < y$ in A. Denote the series subscript of x by $n_1 n_2 \cdots n_\alpha$, and the series subscript of y by $n_1 n_2 \cdots n_\alpha n_{\alpha+1}$. Let d be any element of D. Denote its series subscript by $n'_1 n'_2 \cdots n'_\beta$. We have $\beta < \alpha$. Now suppose $x < d$ in L. Since $n_1 n_2 \cdots n_\alpha$ cannot be a lower segment of $n'_1 n'_2 \cdots n'_\beta$, there will exist a $\mu \leqq \beta$ such that $n_\lambda = n'_\lambda$ for $\lambda < \mu$ and $n_\mu < n'_\mu$. Since the series subscript of x is a lower segment of the series subscript of y, it is clear that we shall have $y < d$ in L. In other words, d cannot lie between x and y, and (**) is verified.

We now prove (***). Let p and q be any two elements of L such that $p < q$ in L. We shall show first that there exists a z in L such that $p < z$ in A, $z < q$ in L, but $z \phi q$ in A. Denote the series subscript of p by $m_1 m_2 \cdots m_\mu$ and the series subscript of q by $n_1 n_2 \cdots n_\nu$.

Case 1. $m_1 m_2 \cdots m_\mu$ is a lower segment of $n_1 n_2 \cdots n_\nu$.

In this case we may choose for z an element whose series subscript is of the form $m_1 m_2 \cdots m_\mu m_{\mu+1}$, where $m_{\mu+1} < n_{\mu+1}$.

Case 2. There exists an ordinal λ such that $\lambda \leqq \mu$, $\lambda \leqq \nu$, $m_\beta = n_\beta$ for $\beta < \lambda$, and $m_\lambda < n_\lambda$.

In this case we may choose for z any element whose series subscript is of the form $m_1 m_2 \cdots m_\mu m_{\mu+1}$.

Now let (p', q') and (p'', q'') be non-overlapping intervals of L. There exist elements z' and z'' of L such that $p' < z'$ and $p'' < z''$ in A, $z' < q'$ and $z'' < q''$ in L, but $z' \phi q'$ and $z'' \phi q''$ in A. We must have $z' \phi z''$ in A. For assume the contrary. We may assume $z' < z''$ in A. Then the series subscript of z' is a lower segment of the series subscript of z''. But $z' < q'$ in L, and $z' \phi q'$ in A. It follows easily that $z'' < q'$ in L. We have, then, $p' < z' < z'' < q'$ in L. But this is impossible since (p', q') and (p'', q'') do not

overlap. It is now clear that a nondenumerable infinity of non-overlapping intervals would give rise to a nondenumerable infinity of mutually incomparable elements, z, of A. But this contradicts VI.

Now it may be that L is not continuous. However, the argument showing that L has property (***) shows also that no element of L has an immediate successor. Hence L contains no "jumps." Accordingly we can fill in the voids (by adding new elements to L) so as to obtain a continuous linear order $\mathfrak{L}$.

Obviously (*) still holds for $\mathfrak{L}$.

Furthermore, between any two elements of $\mathfrak{L}$ there will be two elements of L. Hence (***) holds for $\mathfrak{L}$.

Moreover, it is easily shown that (**) holds for $\mathfrak{L}$, since (**) holds for L and (***) holds for $\mathfrak{L}$.

THE UNIVERSITY OF MICHIGAN.

THE BEHAVIOR OF THE KERNEL FUNCTION AT BOUNDARY POINTS OF THE SECOND ORDER.*

By STEFAN BERGMAN.

1. Introduction. A large chapter of the theory of functions is devoted to the study of the behavior of analytic functions on the boundary of their domains of definition. In the case of functions of one variable and of sufficiently regular domains, say, of domains each bounded by a rectifiable Jordan curve, no necessity arises for considering different classes of boundary points.

We meet quite a different situation in the case of analytic functions of two complex variables. The latter are defined in four dimensional domains (domains of the space of two complex variables). There is no analogue of Riemann's mapping theorem and, as we shall see, there exist essentially different classes of boundary points.

A two-dimensional simply connected domain is completely characterized by the analytic function $z^* = f(z, t)$ which maps it conformally into a circle, taking the point t into the origin, and possessing at this point the derivative 1 (mapping function). A domain, $\mathfrak{D}$, of the space of two complex variables can be characterized to a certain extent by the so-called kernel function, $K_{\mathfrak{D}}(z_1, z_2; \bar{t}_1, \bar{t}_2)$, which can be defined in the following way. Let

$$\{\phi_\nu(z_1, z_2)\}, \quad (\nu = 1, 2, 3, \cdots,) \qquad z_k = x_k + iy_k,$$

be a closed system of orthonormal functions in the domain considered, i. e. a sequence of regular analytic functions with the property that

$$(1.1) \qquad \iint_{\mathfrak{D}} \phi_\nu(z_1, z_2)\overline{\phi_\mu(z_1, z_2)}\, d\omega = 1 \text{ for } \nu = \mu$$

$$= 0 \text{ for } \nu \neq \mu, \quad (\nu, \mu = 1, 2, 3, \cdots),$$

$$d\omega = dx_1 dy_1 dx_2 dy_2,$$

and such that the relation

$$(1.2) \qquad \sum_{\nu=1}^{\infty} \left| \iint_{\mathfrak{D}} f(z_1, z_2)\overline{\phi_\nu(z_1, z_2)}\, d\omega \right|^2 = \iint_{\mathfrak{D}} |f(z_1, z_2)|^2\, d\omega$$

* Received May 16, 1942.

holds for every analytic function $f(z_1, z_2)$ defined in $\mathfrak{D}$ for which the last integral is finite.[1] Then [2]

$$K_{\mathfrak{D}}(z_1, z_2; \bar{t}_1, \bar{t}_2) = \sum_{\nu=1}^{\infty} \phi_\nu(z_1, z_2)\overline{\phi_\nu(t_1, t_2)}. \tag{1.3}$$

([1], § 1).

We shall classify boundary points of a four-dimensional domain according to the behavior of the kernel function at these points. It appears that this classification determines to a certain extent the behavior of any analytic function at these points. In the following we shall use two theorems which characterize the dependence of the kernel function upon the domain.

THEOREM A. *If a PT (pseudo-conformal transformation) i. e. a one-to-one transformation by a pair of analytic functions,*

$$z^*_1 = w_1(z_1, z_2), \qquad z^*_2 = w_2(z_1, z_2), \tag{1.4}$$

maps the domain $\mathfrak{D}$ into $\mathfrak{D}^*$, then

$$K_{\mathfrak{D}}(z_1, z_2; \bar{z}_1, \bar{z}_2) = K_{\mathfrak{D}^*}(z^*_1, z^*_2; \bar{z}^*_1, \bar{z}^*_2) \,|\, \partial(z^*_1, z^*_2)/\partial(z_1, z_2)|^2. \tag{1.5}$$

THEOREM B. *If the domain $\mathfrak{D}$ is contained in the domain $\mathfrak{A}$ ($\mathfrak{D} \subset \mathfrak{A}$) then*

$$K_{\mathfrak{D}}(z_1, z_2; \bar{z}_1, \bar{z}_2) \geqq K_{\mathfrak{A}}(z_1, z_2; \bar{z}_1, \bar{z}_2). \tag{1.6}$$

(See [1], § 1; [3], p. 100 and V; [4], IV).

Remark. The relation (1.5) enables us to define the kernel function for non-schlicht domains which can be transformed pseudo-conformally into schlicht ones.

Theorem B enables us to give bounds for the kernel function of a domain with the aid of *standard domains of comparison*; i. e., by introducing simpler domains which either contain the domain considered or are contained in it, and for which the kernel function can be easily calculated or at least estimated.

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[1] The existence of systems $\{\phi_\nu(z_1, z_2)\}$ can be proved for a large class of domains, in particular for all bounded domains ([4], p. 28).

The numbers in brackets refer to the bibliography p. 700. The Roman number indicates the chapter of the corresponding paper.

For the convenience of the reader we often indicate several places in which the proof or the explanations of the results used are given, so that our indications partially overlap.

[2] The series (1.3) converges uniformly in every closed sub-domain of $\mathfrak{D}$. The kernel function is therefore a regular function of $(z_1, z_2) \in \mathfrak{D}$, $(t_1, t_2) \in \mathfrak{D}$ ([4] p. 23).

Of course, the kernel function can be defined also for two dimensional domains $\mathfrak{B}^2$. If $\mathfrak{B}^2$ is simply connected, the kernel function is connected with the mapping function by a simple relation. See [4], p. 33.

In order to illustrate the basic idea of the method used, let us apply it to the case of a domain of one complex variable. Suppose $\mathfrak{B}^2$ is a bounded domain of the z-plane which we suppose for the sake of simplicity to be convex. We assume further that the radius of curvature of the boundary $\mathfrak{C}^1$ of $\mathfrak{B}^2$ in the neighborhood of the boundary point Q is positive. Thus we can inscribe a circle $\mathfrak{T}^2$, $\mathfrak{T}^2 \subset \mathfrak{B}^2$, whose boundary curve passes through Q and has a common tangent with $\mathfrak{C}^1$ at Q. We assume that Q is the origin and that the inner normal at Q coincides with the positive x-axis. The half plane $\mathfrak{A}^2 = \mathrm{E}[x > 0]$ will obviously contain $\mathfrak{B}^2$. ($\mathfrak{T}^2$ is an interior and $\mathfrak{A}^2$ an exterior domain of comparison). Since $\mathfrak{T}^2 \subset \mathfrak{B}^2 \subset \mathfrak{A}^2$ we have

$$(z+\bar{z})^2 \mathrm{K}_{\mathfrak{A}^2}(z,\bar{z}) \leqq (z+\bar{z})^2 \mathrm{K}_{\mathfrak{B}^2}(z,\bar{z}) \leqq (z+\bar{z})^2 \mathrm{K}_{\mathfrak{T}^2}(z,\bar{z}).$$

Since

$$\mathrm{K}_{\mathfrak{A}^2}(z,\bar{z}) = \pi^{-1}(z+\bar{z})^{-2}, \qquad \mathrm{K}_{\mathfrak{T}^2}(z,\bar{z}) = \pi^{-1}(z+\bar{z}+|z|^2/r)^{-2}$$

(r being the radius of $\mathfrak{T}^2$) we have

$$\lim_{z\to 0}(z+\bar{z})\mathrm{K}_{\mathfrak{A}^2}(z,\bar{z}) = \lim_{z\to 0}(z+\bar{z})^2 \mathrm{K}_{\mathfrak{T}^2}(z,\bar{z}) = \pi^{-1},$$

provided that we approach Q in the sector $[2|z|/(z+\bar{z})] \leqq 1/\cos\alpha$, $|\alpha| < \pi/2$. It follows that

$$\lim_{z\to 0}(z+\bar{z})^2 \mathrm{K}_{\mathfrak{B}^2}(z,\bar{z}) = 1/\pi.$$

Notation. Manifolds will be denoted by German letters, the upper index indicating the dimension of the manifold. However, we omit this upper index in the case of four-dimensional manifolds.

In operating with sets we use the usual symbols: S, $+$ (sumset), $-$ (difference set), $\cdot$ (intersection), $\times$ (topological product of domains) etc.[3] E. g. we denote by $\mathsf{S}_{\gamma \in \mathfrak{R}^m} \mathfrak{C}^n(\gamma)$ the sumset of a family of sets which depends upon a parameter γ running through the region $\mathfrak{R}^m$. (Note that the sets $\mathfrak{C}^n(\gamma)$ considered in this paper are frequently families of disjoint manifolds lying in the four-dimensional space $[\mathfrak{C}^n(\gamma)\cdot\mathfrak{C}^n(\gamma') = 0$, for $\gamma \neq \gamma']$ so that $\mathsf{S}_{\gamma \in \mathfrak{R}^m} \mathfrak{C}^n(\gamma)$ has the dimension $n+m$).

We denote by E (...) the set of points whose coordinates satisfy the relations indicated in the brackets.

2. The order of a boundary point. One of the methods of classifying the boundary points of a four-dimensional domain is by establishing con-

[3] See F. Hausdorff, *Lehrbuch der Mengenlehre*, 2nd ed., 1927, p. 1.

nections between the geometrical properties of the immediate neighborhood of the boundary points and the behavior of the kernel function at these points. Evidently, the geometry considered cannot be simply the Euclidean four-dimensional geometry, but must be the geometry of the space of two complex variables, i. e. the geometry in which analytic[4] planes, surfaces and hypersurfaces are distinguished from ordinary planes, surfaces and hypersurfaces. ([3], I).

A rough geometrical classification of the boundary points is achieved in the following way. Let $Q = \{Z_1, Z_2\}$ be a boundary point of the four-dimensional domain $\mathfrak{D}$. We suppose at first that the boundary of $\mathfrak{D}$ is a hypersurface which possesses at the point Q an analytic tangential plane, or at least that Q is a point of intersection of several hypersurfaces possessing at Q analytic tangential planes. We consider all *analytic* surfaces which pass through Q. The following cases may occur:

(a) There exists one surface which has no points in common with the boundary of $\mathfrak{D}$, except of course the point Q.

(b) There exists a surface a segment of which lies in the boundary of $\mathfrak{D}$. Q is an interior point of this segment.

(c) There exist two or more surfaces which have segments in common with the boundary. Q is the point of intersection of these surfaces and belongs to the boundary of one of the above segments. (We may call this intersection an "edge").

(d) There exists one surface a segment of which lies entirely inside of the domain $\mathfrak{D}$, except for the point Q. Q is an interior point of this segment.

Note that these properties are invariant with respect to these PT's which are defined in a domain containing $\mathfrak{D}$. At first it might seem that the same boundary point could possess two or more of the properties indicated above. We shall see that under certain conditions this can not happen.

We shall now classify the boundary points with the aid of the kernel function. If the point (z_1, z_2) approaches the boundary point Q, the kernel function $\mathbf{K}_{\mathfrak{D}}(z_1, z_2, \bar{z}_1, \bar{z}_2)$ will, in the general case, become infinite. If the expression

$$(2.1) \qquad \left| \sqrt{|Z_1 - z_1|^2 + |Z_2 - z_2|^2} \right|^{\tau} \mathbf{K}_{\mathfrak{D}}(z_1, z_2, \bar{z}_1, \bar{z}_2)$$

[4] In order to avoid any ambiguity we note that the words "analytic plane," "analytic surface," etc. are to be understood in the usual meaning of these terms in the theory of functions of several complex variables.

remains between two positive bounds when $\{z_1, z_2\}$ approaches $Q = (Z_1, Z_2)$ in a manner which may be subjected to certain restrictions, we shall call Q an R_τ-point (R-point of the order τ). If furthermore, the expression (2.1) possesses a definite positive limit provided that $\lim \arg (Z_2 - z_2)$, $\lim \arg (Z_1 - z_1)$ and $\lim |(Z_2 - z_2)/(Z_1 - z_1)|$ ($\neq 0$ or ∞) exist, we shall call Q an L_τ-point (L-point of the order τ). Note that the restrictions concerning the manner in which (z_1, z_2) approaches Q may vary according to τ and that numerous other subdivisions are possible.

If certain additional hypotheses are made, a boundary point possessing one of the properties (a)-(d) is an R-point of the order τ and [4]

$\tau = 3$ in the case (a), $\tau = 4$ in the case (c),

$\tau = 2$ in the case (b), $\tau = 0$ in the case (d).

This can be relatively easily understood by considering simple examples. (See [1], p. 17). The precise formulations of sufficient conditions for a boundary point to be an R-point or an L-point of the order τ requires, however, a more thorough investigation of the geometrical properties of the boundary. For boundary points of the third order this has been done in [1], § 5.[5] In this paper we shall concern ourselves with the points of the second order. The method used will be that of domains of comparison.[6]

3. Standard domains for points of the second order. Consider a domain $\mathfrak{S}$ possessing the following three properties:

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[5] Formulating the sufficient conditions to insure that a point Q is a limit point of the third order, we supposed in [1], p. 11, that there exists a one-to-one transformation of the form

$$z_1 = z^*_1 + 2bz^*_1 z^*_2 + cz^{*2}_2 + \cdots, \qquad z_2 = z^*_2$$

so that the equation of the boundary hypersurface (in a sufficiently small neighborhood) can be written in the form $2x^*_1 = ay^{*2}_1 + \sigma |z^*_2|^2 + \cdots$ [See [1], (3.17) and (3.18)]. We note that if the projection $\mathfrak{P}^2$ of the domain on the z_2 plane omits two points say a_1 and a_2, which satisfy the equation $a_1 + a_2 = -2ba_1a_2$, then such a transformation exists. The *PT*

$$z_1 = z^*_1 + 2bz^*_1 z^*_2 + cz^{*2}_2 + dz^*_1 z^{*2}_2, \quad z_2 = z^*_2, \quad d = 1/a_1a_2,$$

transforms the expression (3.13) of [1] into $2x^*_1 = ay^{*2}_1 + \sigma |z^*_2|^2 + \cdots$ and is one-to-one in the domain, for $1 - 2bz_2 + dz_2^2 = d(z_2 - a_1)(z_2 - a_2) \neq 0$, $z_2 \in \mathfrak{P}^2$.

[6] The points of the order 0 are "points of non existence": i.e. every analytic function regular in the interior of the domain is, in general, regular also at such boundary points. Points of the fourth order appear in particular in domains possessing a distinguished boundary surface; all points of this surface are, in general, of the fourth order.

(α) The point [7] $Q(0, Z_2)$ is a boundary point of $\mathfrak{S}$;

(β) $\mathfrak{S}$ is invariant under the transformation

$$z'_1 = \mu z_1, \; z'_2 = z_2 \text{ for every } \mu > 0;$$

(γ) there is a sequence of points $\{z_1^{(\nu)}, z_2^{(\nu)}\}$ in $\mathfrak{S}$ approaching Q and such that

$$\phi \equiv \lim_{\nu \to \infty} \arg z_1^{(\nu)} \text{ exists, } \lim_{\nu \to \infty} \left| \frac{z_1^{(\nu)}}{z_2^{(\nu)} - Z_2} \right| \text{ exists, } (e^{i\phi}, Z_2) \in \mathfrak{S}.$$

For such a domain $\mathfrak{S}$ we have by (β) and Theorem A

$$(3.1) \quad |z_1^{(\nu)}|^2 K_{\mathfrak{S}}(z_1^{(\nu)}, z_2^{(\nu)}; \overline{z_1^{(\nu)}}, \overline{z_2^{(\nu)}}) = K_{\mathfrak{S}}(e^{i\phi_\nu}, z_2^{(\nu)}; e^{i\phi_\nu}, \overline{z_2^{(\nu)}}), \quad \phi_\nu = \arg z_1^{(\nu)}.$$

Next, by (γ), the point $(e^{i\phi}, Z_2)$ lies in $\mathfrak{S}$ and hence by continuity of K the expression (3.1) approaches a limit as $\nu \to \infty$. In view of the existence of the second limit in (γ) this means that $Q(0, Z_2)$ is an L-point of order two for the sequence $\{z_1^{(\nu)}, z_2^{(\nu)}\}$.

This example shows us how to find conditions for points of the second order. We shall construct domains $\mathfrak{S}$ possessing the above properties and use domains obtained from $\mathfrak{S}$ by suitable PT's as domains of comparison.

Let r be a positive number and ϑ_1 and ϑ_2 two real numbers such as $0 < \vartheta_2 - \vartheta_1 < 2\pi$. By $\mathfrak{S}^2(\vartheta_1, \vartheta_2, r)$ we shall denote the angular domain $E[0 < |z| < r, \; \vartheta_1 < \arg z < \vartheta_2]$, by $\mathfrak{S}^2(\vartheta_1, \vartheta_2)$ the angular domain $E[0 < |z| < \infty, \; \vartheta_1 < \arg z < \vartheta_2]$.

Suppose that $\mathfrak{H}^2$ is a simply connected domain of the γ-plane bounded by a rectifiable Jordan curve, and that $\vartheta_1(\gamma)$ and $\vartheta_2(\gamma)$ are two real valued functions defined in $\mathfrak{H}^2$ satisfying the following conditions:

1. $\vartheta_1(\gamma)$ and $\vartheta_2(\gamma)$ are uniformly continuous and continuously differentiable.

2. There exists a positive constant ω such that

$$(3.1) \qquad \omega < \vartheta_2(\gamma) - \vartheta_1(\gamma) < \pi.$$

3. There exist four real constants, $\vartheta_1^0, \vartheta_2^0, \vartheta_1^1, \vartheta_2^1$, such that

$$(3.2) \qquad -\pi/2 < \vartheta_1^1 < 0 < \vartheta_2^1 < \pi/2 < \vartheta_1^0 < \pi < \vartheta_2^0 < 3\pi/2,$$

$$(3.3) \qquad \vartheta_2^0 - 2\pi < \vartheta_1(\gamma) < \vartheta_1^1, \qquad \vartheta_2^1 < \vartheta_2(\gamma) < \vartheta_1^0.$$

We may now consider the four dimensional domain $\mathfrak{S}$ given by

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[7] For convenience we take the boundary point in question to be the point $Q(0, Z_2)$.

$$\mathfrak{S} = \sum_{\gamma \in \mathfrak{N}^m} E[z_2 = \gamma, z_1 \in \mathfrak{S}^2(\vartheta_1(\gamma), \vartheta_2(\gamma))]. \tag{3.4}$$

Such a domain we shall call a standard domain of the $\mathfrak{S}$-type (See [1], § 7).

Our next task consists in investigating the behavior of the kernel function $K_{\mathfrak{S}}(z_1, z_2; \bar{z}_1, \bar{z}_2)$ of our standard domain in the neighborhood of the boundary points $\{0, Z_2\}$, $Z_2 \in \mathfrak{H}^2$. This can be done without the actual calculation of the kernel function.

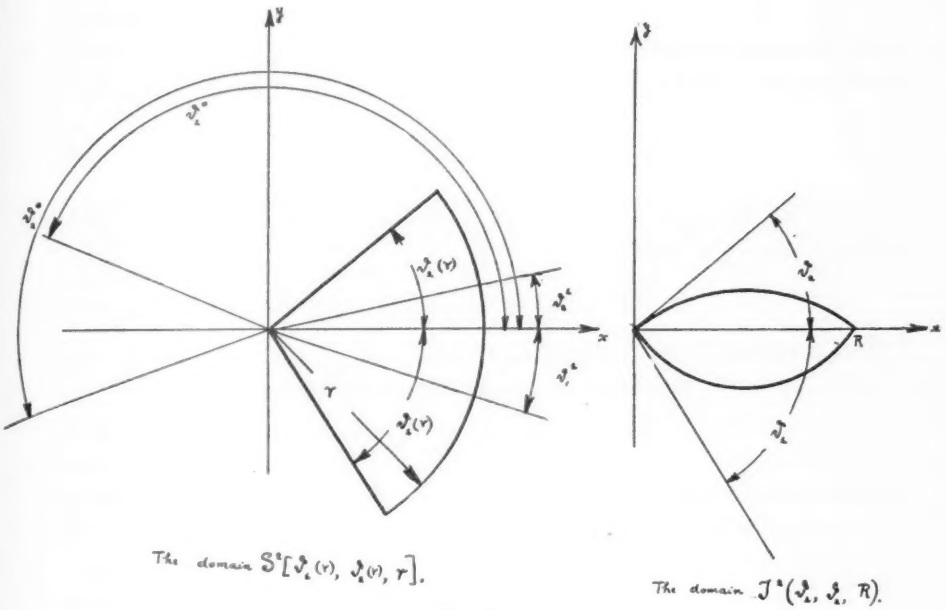

Fig. 1.

We first introduce the concept *A-approach with respect to a standard domain of the* $\mathfrak{S}$*-type*. Consider a given $\mathfrak{S}$-domain and a sequence of points $\{z_1^{(\nu)}, z_2^{(\nu)}\}$ situated in this domain and such that

$$\{z_1^{(\nu)}, z_2^{(\nu)}\} \to (0, Z_2), \qquad \nu \to \infty. \tag{3.5}$$

If there exists a positive ϵ and a closed subdomain $\mathfrak{H}_0^2 \subset \mathfrak{H}^2$ such that

$$(z_1^{(\nu)}, z_2^{(\nu)}) \in \sum_{\gamma \in \mathfrak{H}_0^2} E[z_2 = \gamma,\ z_1 \in \mathfrak{S}^2(\vartheta_1(\gamma) + \epsilon,\ \vartheta_2(\gamma) - \epsilon)], \quad (\nu = 1, 2, \cdots), \tag{3.6}$$

we call the approach (3.5) a sectorial one. If furthermore,

$$\lim_{\nu \to \infty} \arg z_1^{(\nu)} \quad \text{and} \quad \lim_{\nu \to \infty} |z_1^{(\nu)}/(z_2^{(\nu)} - Z_2)| \tag{3.7}$$

exist, we call the approach (3.6) an A-approach.[8]

[8] In [1], p. 29, we denoted this approach as $AII(\alpha)$-approach, $\alpha = \lim_{\nu \to \infty} \arg z_1^{(\nu)}$.

Consider now an arbitrary four-dimensional domain $\mathfrak{D}$. Let $(0, Z_2)$ be a boundary point of $\mathfrak{D}$ and suppose that the part of the boundary situated in a sufficiently small neighborhood of $(0, Z_2)$ contains a part of the analytic plane $z_1 = 0$, namely $\mathrm{E}[z_1 = 0,\ z_2 \epsilon \mathfrak{B}^2]$, where $\mathfrak{B}^2$ is a simply connected domain in the z_2-plane containing the point $z_2 = Z_2$. Let $\phi(z_1, z_2)$ be a function defined in $\mathfrak{D}$. We shall say that this function possesses A-limits at the point $(0, Z_2)$ if

$$\lim_{\nu\to\infty} \phi(z_1^{(\nu)}, z_2^{(\nu)}) \tag{3.8}$$

exists for every sequence of points $(z_1^{(\nu)}, z_2^{(\nu)})$, $(z_1^{(\nu)}, z_2^{(\nu)}) \epsilon \mathfrak{D}$, $(\nu = 1, 2, \cdots)$, satisfying the following conditions:

1.

$$\{z_1^{(\nu)}, z_2^{(\nu)}\} \to (0, Z_2), \qquad \nu \to \infty. \tag{3.9}$$

2. The approach (3.9) is an A-approach with respect to a certain standard domain $\mathfrak{S}$ for which

$$\mathfrak{S} \cdot \mathrm{E}[\,|\,z_1\,| < \delta, \qquad |\,z_2\,| < \infty\,] \subset \mathfrak{D},$$

δ being a sufficiently small positive number.

We shall now show that *for an $\mathfrak{S}$-domain the expression*

$$|\,z_1\,|^2 \mathrm{K}_{\mathfrak{S}}(z_1, z_2;\ \bar{z}_1, \bar{z}_2) \tag{3.11}$$

possesses at any boundary point $(0, Z_2)$, $Z_2 \epsilon \mathfrak{B}^2$, *positive A-limits.* To this end we apply to the domain $\mathfrak{S}$ the PT

$$\tilde{z}_1 = \mu z_1, \quad \tilde{z}_2 = z_2 \qquad (\mu > 0). \tag{3.12}$$

Since this PT maps $\mathfrak{S}$ into itself we have by Theorem A (see **Introduction**)

$$|\,z_1\,|^2 \mathrm{K}_{\mathfrak{S}}(z_1, z_2;\ \bar{z}_1, z_2) = |\,z_1/\mu\,|^2 \mathrm{K}_{\mathfrak{S}}(z_1/\mu, z_2;\ \bar{z}_1/\mu, \bar{z}_2). \tag{3.13}$$

This formula shows that

$$r^2 \mathrm{K}_{\mathfrak{S}}(re^{i\phi}, z_2;\ re^{-i\phi}, \bar{z}_2) \tag{3.14}$$

is a function of z_2 and ϕ only. This function is continuous, for the kernel function is regular in the interior of the domain. Clearly, it is positive. Therefore the limit

$$\lim_{\substack{r\to 0 \\ \phi\to\Phi \\ z_2\to Z_2}} r^2 \mathrm{K}_{\mathfrak{S}}(re^{i\phi}, z_2, re^{-i\phi}, z_2) \tag{3.15}$$

exists, provided that the approach $(re^{i\phi}, z_2) \to (0, Z_2)$ is an A-approach. Thus our assertion is proved.

Consider the *PT*

$$z^*_1 = z_1/(1 + R^{-1}z_1), \qquad z^*_2 = z_2, \qquad (R \text{ real}). \tag{3.20}$$

It transforms $\mathfrak{S}$ into a domain we shall denote by $\mathfrak{J}$. The linear transformation

$$z^*_1 = z_1/(1 + R^{-1}z_1) \tag{3.21}$$

maps $\mathfrak{S}^2(\vartheta_1, \vartheta_2)$ into the domain $\mathfrak{J}^2(\vartheta_1, \vartheta_2, R)$ which is bounded by two circular arcs passing through the origin and through the point R and forming with the real axis the angles ϑ_1 and ϑ_2, respectively. [Clearly, $\mathfrak{J}^2(\vartheta_1, \vartheta_2, R) \subset \mathfrak{S}^2(\vartheta_1, \vartheta_2, R)$]. Hence, $\mathfrak{J}$ can be represented in the form

$$\mathfrak{J} = \mathop{\mathsf{S}}_{\gamma \,\epsilon\, \mathfrak{H}^2} \mathrm{E}[z_2 = \gamma,\ z_1 \,\epsilon\, \mathfrak{J}^2(\vartheta_1(\gamma), \vartheta_2(\gamma), R)].$$

Such a domain we shall call standard domain of the $\mathfrak{J}$-type. Its kernel function possesses at any point $(0, Z_2)$ positive A-limits. This follows immediately from Theorem A, for the Jacobian of the transformation (3.20) is equal to 1 at every point of the analytic plane $z_1 = 0$.

4. *R*-points of the second order. In this and the following sections we shall consider a domain $\mathfrak{D}$ which possesses the property (b) (See p. 682). That is to say, we suppose that there exists an analytic surface $\mathfrak{F}^2$ which possesses a segment in common with the boundary of D. $Q = (Z_1, Z_2)$ is an interior point of this segment. We shall find sufficient additional conditions to insure that Q be a point of the second order.

Let $\mathfrak{P}_2{}^2(\mathfrak{D})$ be the projection of $\mathfrak{D}$ on the plane $z_1 = 0$; i. e. the set of all points z_2, such that for suitably chosen z_1, $(z_1, z_2) \,\epsilon\, \mathfrak{D}$. We suppose the following hypotheses to be fulfilled.

Hypothesis 4. 1. The analytic surface $\mathfrak{F}^2$ may be represented in the form

$$z_1 = h(z_2), \qquad Z_1 = h(Z_2), \tag{4.1}$$

$h(z_2)$ being an analytic function which is regular and univalent in $\mathfrak{P}_2{}^2(\mathfrak{D})$.

The *PT*

$$z^*_1 = z_1 - h(z_2), \qquad z^*_2 = z_2 - Z_2 \tag{4.2}$$

takes $\mathfrak{D}$ into another domain whose boundary contains a segment, $\mathfrak{H}^2$, of the analytic plane $z_1 = 0$. Q is taken into the point $(0, 0)$. We may consider the coordinates z^*_1, z^*_2 to have been introduced from the beginning and replace hypothesis 4. 1 by

Hypothesis 4. 1a. The boundary of $\mathfrak{D}$ contains a segment, $\mathfrak{H}^2$, of the analytic plane $z_1 = 0$, $0 \,\epsilon\, \mathfrak{H}^2$.

Hypothesis 4. 2. There exist two positive constants ρ and α, $\alpha < \pi/2$, a regular harmonic function $\omega(\gamma)$ defined in a simply connected domain $\mathfrak{E}^2 \supset \mathfrak{P}_2{}^2(\mathfrak{D})$, and a closed subdomain $\bar{\mathfrak{W}}^2$ of $\mathfrak{H}^2$, $0 \in \mathfrak{H}^2$, such that

$$(4.3) \qquad \mathfrak{S}^2[\omega(\gamma) - \alpha, \omega(\gamma) + \alpha, \rho] \subset \mathfrak{D} \cdot \mathrm{E}(z_2 = \gamma), \qquad \gamma \in \mathfrak{W}^2,$$

and

$$(4.4) \qquad \mathfrak{S}^2[-\omega(\gamma) - \pi + \alpha, \omega(\gamma) + \pi - \alpha] \supset \mathfrak{D} \cdot \mathrm{E}(z_2 = \gamma), \qquad \gamma \in \mathfrak{E}^2.$$

Let $\psi(\gamma)$ be the conjugate harmonic function to $\omega(\gamma)$. We may suppose that $\psi(\gamma)$ does not vanish in $\mathfrak{E}^2$. The *PT*

$$(4.5) \qquad z^*{}_1 = z_1 e^{\psi(\gamma) - i\omega(\gamma)}, \qquad z^*{}_2 = z_2$$

maps $\mathfrak{D}$ into a domain which satisfies hypothesis 4. 1a. The direction $\omega(\gamma)$ is taken into the positive x_1-direction. We call $z^*{}_1, z^*{}_2$ the *normal co-ordinates with respect to the boundary point Q, the analytic surface $\mathfrak{H}^2$ and the harmonic function $\omega(\gamma)$*. We may suppose that the normal co-ordinates have been introduced from the beginning and replace hypothesis 4. 2 by the following

Hypothesis 4. 2a. There exists a simply connected domain $\mathfrak{E}^2 \supset \mathfrak{P}_2{}^2(\mathfrak{D})$ and a closed subdomain $\bar{\mathfrak{W}}^2$ of $\mathfrak{H}^2$, which contains 0 as an interior point, such that

$$(4.6) \qquad \mathfrak{S}^2(-\alpha, \alpha, \rho) \subset \mathfrak{D} \cdot \mathrm{E}(z_2 - \gamma), \qquad \gamma \in \mathfrak{W}^2,$$

and

$$(4.7) \qquad \mathfrak{S}^2(-\pi + \alpha, \pi - \alpha) \supset \mathfrak{D} \cdot \mathrm{E}(z_2 = \gamma), \qquad \gamma \in \mathfrak{E}^2.$$

If our hypotheses are satisfied, the domain $\mathfrak{D}$ belongs to the class for which we defined the concept of A-approach. We shall prove

THEOREM I. *If $\mathfrak{D}$ satisfies the hypotheses* 4. 1 *and* 4. 2, $Q = (0, 0)$ *is an R-point of the second order.*

Proof. We must show that the expression

$$(4.8) \qquad |z_1{}^{(\nu)}|^2 \mathrm{K}_{\mathfrak{D}}(z_1{}^{(\nu)}, z_2{}^{(\nu)}; \bar{z}_1{}^{(\nu)}, \bar{z}_2{}^{(\nu)})$$

remains between two positive bounds as the point $(z_1{}^{(\nu)}, z_2{}^{(\nu)})$ approaches our boundary point $(0, 0)$ in the way defined in **3** (A-approach). By hypothesis 4. 2a the $\mathfrak{J}$-domain

$$(4.9) \qquad \mathfrak{J} = \sum_{\gamma \in \mathfrak{W}^2} \mathrm{E}[z_2 = \gamma, z_1 \in \mathfrak{J}^2(-\alpha, \alpha, \rho)]$$

is an interior, and the $\mathfrak{S}$-domain

$$\mathfrak{A} = \sum_{\gamma \in \mathfrak{E}^2} E[z_2 = \gamma, z_1 \in \mathfrak{S}^2(-\pi + \alpha, \pi - \alpha)] \tag{4.10}$$

an exterior domain of comparison for $\mathfrak{D}$:

$$\mathfrak{J} \subset \mathfrak{D} \subset \mathfrak{A}. \tag{4.11}$$

For sufficiently large ν, the points $(z_1^{(\nu)}, z_2^{(\nu)})$ are situated within $\mathfrak{J}$. We have then by Theorem B

$$|z_1^{(\nu)}|^2 K_{\mathfrak{A}}(z_1^{(\nu)}, z_2^{(\nu)}; z_1^{(\nu)}, z_2^{(\nu)}) \leqq |z_1^{(\nu)}|^2 K_{\mathfrak{D}}(z_1^{(\nu)}, z_2^{(\nu)}; z_1^{(\nu)}, z_2^{(\nu)}) \leqq |z_1^{(\nu)}|^2 K_{\mathfrak{J}}(z_1^{(\nu)}, z_2^{(\nu)}; z_1^{(\nu)}, z_2^{(\nu)}). \tag{4.12}$$

In **3** we proved that the expressions on the right and left hand side in (4.12) possess non-vanishing limits as $\nu \to \infty$. This proves our theorem.

5. Sufficient conditions for *L*-points of second order. In this section we shall formulate conditions which are sufficient to insure that a boundary point is an L-point of second order (L_2). We suppose that our domain $\mathfrak{D}$ satisfies the hypotheses 4.1a and 4.2a.

Hypotheses 5.1. $\mathfrak{S}^2$ is a convex domain.

We write the equation of its boundary curve in the form

$$z_2 = R(\phi)e^{i\phi}, \qquad 0 \leqq \phi \leqq 2\pi. \tag{5.1}$$

Let k be a positive number. We shall denote by $\mathfrak{S}^2(1+k)$ the domain bounded by the curve

$$z_2 = (1+k)Re^{i\phi}, \qquad 0 \leqq \phi \leqq 2\pi. \tag{5.2}$$

A convex domain possesses the following simple property:

LEMMA 1. *For every positive k there exists a positive ϵ, such that all points, ζ, which may be represented in the form*

$$\zeta = e^{i\xi}z_2, \quad z_2 \in \mathfrak{S}^2(1+k), \quad -\epsilon \leqq \xi \leqq \epsilon, \tag{5.3}$$

are situated within $\mathfrak{S}^2(1+2k)$.

Hypothesis 5.2. $\mathfrak{D}$ is a bounded domain.

For the sake of simplicity we may suppose that this hypothesis is replaced by

Hypothesis 5.2a. If $(z_1, z_2) \in \mathfrak{D}$, then $|z_1| < 1/2$, $|z_2| < 1$.

Hypothesis 5. 3. There exists a positive function $\kappa(t)$,

$$\lim_{t\to 0} \kappa(t) = 0, \tag{5.4}$$

such that for every point $(z_1, z_2) \in \mathfrak{D}$ with $|z_1| \leqq t$

$$z_2 \in \mathfrak{H}^2[1 + \kappa(t)]. \tag{5.5}$$

Hypothesis 5. 4. There exists a positive constant η_0 such that the intersection

$$\mathfrak{D} \cdot \mathrm{E}[|z_1| < \eta_0] \cdot \mathrm{E}(z_2 = \gamma), \qquad \gamma \in \mathfrak{E}^2, \tag{5.6}$$

(See Hypothesis 4. 2a) is either empty or a simply connected domain which is bounded by two curves,[9] $\alpha_k^1(\gamma)$ starting at the origin and lying in the upper (lower) z_1-half-plane for $k = 2$ $(k = 1)$, and, perhaps, by a part of the circle $|z_1| = \eta_0$ (See Fig. 2).

Hypothesis 5. 5. There exists for every closed sub-domain $\bar{\mathfrak{T}}^2$ of $\mathfrak{H}^2$, $0 \in \mathfrak{T}^2$, a positive constant $\eta_1(\mathfrak{T}^2) < \eta_0$ such that for $\gamma \in \mathfrak{T}^2$ and $|y_1| < \eta_1(\mathfrak{T}^2)$ the curves $\alpha_k^1(\gamma)$ (introduced in Hypothesis 5. 4) may be represented in the form

$$x_1 = A_k(y_1, \gamma), \qquad 0 \leqq (-1)^k y_1, \qquad |y_1| \leqq \eta_1(\mathfrak{T}^2). \tag{5.7}$$

Furthermore, if $\Theta_k(y_1, \gamma) = \cot[\partial A_k(y_1, \gamma)/\partial y_1]$,

$$\theta_k(\gamma) = \Theta_k(0, \gamma). \qquad \gamma \in \mathfrak{H}^2, \tag{5.8}$$

$\Theta_k(y_1, \gamma)$ is a continuously differentiable function of its real variables, and

$$|\operatorname{grad} \theta_k(\gamma)| < C, \tag{5.9}$$

$$\alpha - \pi < \theta_1(\gamma) < -\alpha, \quad \alpha < \theta_2(\gamma) < \pi - \alpha. \tag{5.10}$$

Hypothesis 5. 6. There exist two functions $C_k(y_1, \gamma)$, defined for $|y_1| < \eta_0$, $\gamma \in \bar{\mathfrak{E}}^2$, possessing continuous derivatives with respect to their three real variables, satisfying the conditions

$$C_1(0, \gamma) = C_2(0, \gamma) \tag{5.11}$$

$$\begin{aligned} C_k(0, \gamma) &= 0 \text{ for } \gamma \in \mathfrak{H}^2, \\ &\geqq 0 \text{ for } \gamma \in \mathfrak{E}^2 - \mathfrak{H}^2, \end{aligned} \tag{5.12}$$

and such that for every point $(z_1, z_2) \in \mathfrak{D}$ with $|z_1| < \eta_0$, $z_2 \in \mathfrak{H}^2$,

$$\begin{cases} x_1 > C_2(y_1, z_2) & \text{for } y_1 \geqq 0, \\ x_1 > C_1(y_1, z_2) & \text{for } y_1 \leqq 0. \end{cases} \tag{5.13}$$

---

[9] To avoid repetition, we shall omit the statement "$k = 1, 2$" hereafter. Whenever the index k, occurs, it is to have the values 1 and 2.

(We denote the curves $x_1 = C_k(y_1, \gamma)$ by $\mathfrak{c}_k^1(\gamma)$.

If we write $\theta_k(y_1, \gamma) = \cot [\partial C_k(y_1, \gamma)/\partial y_1]$,

$$\theta_k(\gamma) = \theta_k(0, \gamma), \qquad \gamma \in \mathfrak{E}^2 - \mathfrak{H}^2, \tag{5.14}$$

$\theta_k(y_1, \gamma)$ is still a continuously differentiable function of its three real variables, and the inequalities (5.9) and (5.10) are valid for $\gamma \in \mathfrak{E}^2 - \mathfrak{H}^2$.

The following is an immediate consequence of our hypotheses.

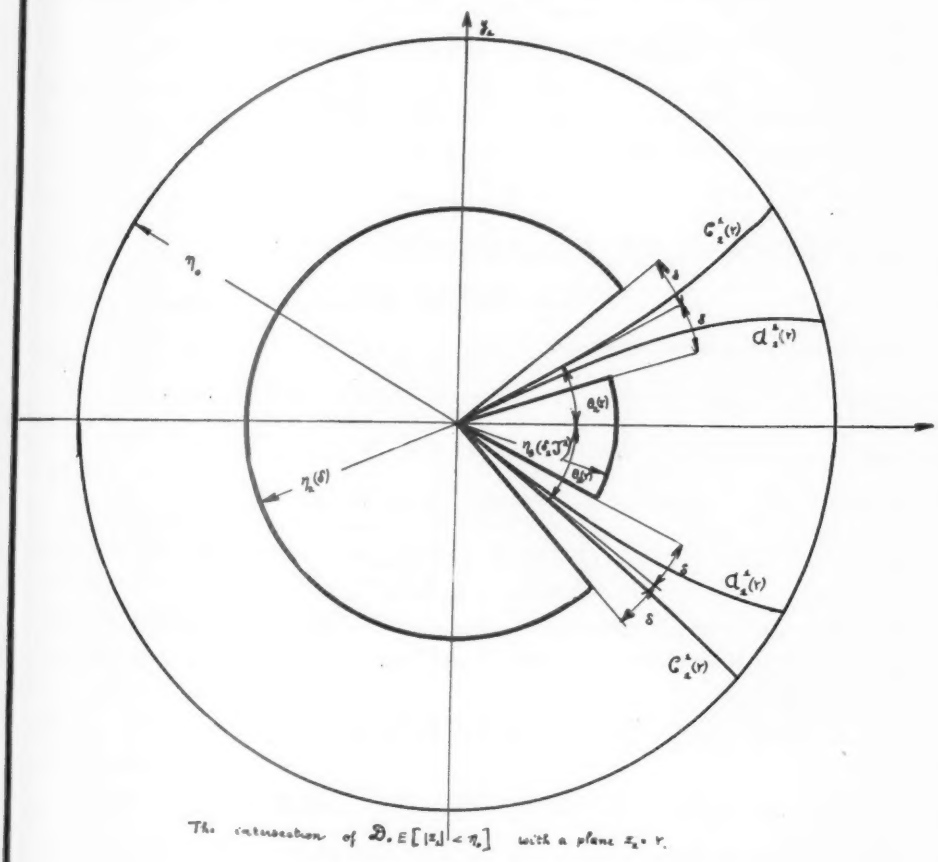

Fig. 2.

LEMMA 2. (A) *For every sufficiently small positive* δ *there exists a positive* $\eta_2(\delta) < \eta_0$,

$$\lim_{\delta \to 0} \eta_2(\delta) = 0 \tag{5.15a}$$

such that

$$\mathfrak{S}^2[\theta_2(\gamma) + \delta, 2\pi + \theta_1(\gamma) - \delta, \eta_2(\delta)] \cdot \mathfrak{D} \cdot \mathrm{E}(z_2 = \gamma) = 0, \quad \gamma \in \mathfrak{E}^2. \tag{5.15}$$

(B) *For every sufficiently small positive* δ, *and for every closed subdomain* $\bar{\mathfrak{T}}^2$ *of* $\mathfrak{S}^2$, $0 \in \mathfrak{T}^2$, *there exists a positive* $\eta_3(\delta, \mathfrak{T}^2) < \eta_1(\mathfrak{T}^2)$ *such that*

$$(5.16) \quad \mathfrak{S}^2[\theta_1(\gamma) + \delta, \theta_2(\gamma) - \delta, \eta_3(\delta, \mathfrak{T}^2)] \subset \mathfrak{D} \cdot \mathrm{E}(z_2 = \gamma), \quad \gamma \in \mathfrak{T}^2.$$

Proof. We choose $\eta_2(\delta)$ so small that

$$(5.17) \quad |\theta_k(\gamma) - \theta_k(\eta, \gamma)| < \delta \text{ for } \gamma \in \mathfrak{E}^2, \ 0 \leqq (-1)^k \eta \leqq (-1)^k \eta_2(\delta).$$

This is possible since $\theta_k(y, \gamma)$ are uniformly continuous. Let $z_1 = x_1 + iy_1$ be a point of

$$(5.18) \qquad \mathfrak{S}^2[\theta_2(\gamma) + \delta, 2\pi + \theta_1(\gamma) - \delta, \eta_2(\delta)].$$

We must show that (z_1, γ) is an exterior point of $\mathfrak{D}$.

It will suffice to show that, say, for $y_1 \geqq 0$,

$$(5.19) \qquad x_1 < C_2(y_1, \gamma)$$

(see Hypothesis 5.6). We have, say, for $\theta_2(\gamma) < \pi/2$,

$$(5.20) \qquad x_1 < y_1 \cot [\theta_2(\gamma) + \delta];$$

and by the mean value theorem,

$$(5.21) \qquad C_2(y_1, \gamma) = C_2(0, \gamma) + y_1 \cot \theta_2(\xi y_1), \quad 0 < \xi < 1.$$

Since $|y_1| < \eta_2(\delta)$ we have by (5.17) and (5.21)

$$(5.22) \qquad C_2(y_1, \gamma) > C_2(0, \gamma) + y_1 \cot [\theta_2(\gamma) + \delta].$$

By (5.20) and (5.22), (5.19) is proved, for $C_2(0, \gamma)$ is non-negative, according to (5.12). The cases $y_1 < 0$, $\theta_2(\gamma) > \pi/2$ etc. may be treated in the same way. Our assertion (A) is thus proved. The proof of (B) is similar. We apply the Hypotheses 5.4 and 5.5 and need only show that if $z_2 \in \mathfrak{T}^2$ and $(z_1, z_2) \in \mathfrak{D}$ then

$$x_1 > A_2(y_1, z_2) \text{ for } 0 \leqq y_1 \leqq \eta_1(\mathfrak{T}^2),$$

and

$$x_1 > A_1(y_1, z_2) \text{ for } -\eta_1(\mathfrak{T}^2) \leqq y_1 \leqq 0,$$

But this follows immediately from the fact that each circle

$$\mathrm{E}[z_2 \in \bar{\mathfrak{T}}^2, |z_1| < \eta_1(\mathfrak{T}^2)]$$

is divided by the curves $\alpha_k^1(z_2)$ into two parts which vary continuously with z_2, one of which (the one containing positive real points z_1) contains inner points of $\mathfrak{D}$.

We shall prove

THEOREM II. *If the domain $\mathfrak{D}$ satisfies the hypotheses* 4. 1-4. 2 *and* 5. 1-5. 6, *the boundary point* $Q = (0, 0)$ *is an L-point of the second order.*

The proof will be given in the following sections.

6. Construction of the domains of comparison for an L_2-point. In order to prove Theorem II we shall construct an interior and an exterior domain of comparison for the domain $\mathfrak{D}$, where $\mathfrak{D}$ satisfies Hypotheses 4. 1-4. 2 and 5. 1-5. 6.

(i) *Interior domain of comparison.* Its construction is possible by virtue of Lemma 2, (B). Let δ be a sufficiently small positive number, $\bar{\mathfrak{T}}^2$ a closed subdomain of $\mathfrak{H}^2$, $0 \epsilon \mathfrak{T}^2$. We write

$$\tau = \eta_3(\delta, \mathfrak{T}^2) \sin \alpha. \tag{6. 1}$$

Then, for $\gamma \epsilon \mathfrak{T}^2$,

$$\mathfrak{Z}^2[\theta_1(\gamma) + \delta, \theta_2(\gamma) - \delta, \tau] \subset \mathfrak{S}^2[\theta_1(\gamma) + \delta, \theta_2(\gamma) - \delta, \eta_3(\delta, \mathfrak{T}^2)], \tag{6. 2}$$

whence it follows by Lemma 2, (B) that

$$\mathfrak{Z} = \sum_{\gamma \epsilon \mathfrak{T}^2} \mathrm{E}[z_2 = \gamma, z_1 \epsilon \mathfrak{Z}^2(\theta_1(\gamma) + \delta, \theta_2(\gamma) - \delta, \tau)] \tag{6. 3}$$

is an interior domain of comparison:

$$\mathfrak{Z} \subset \mathfrak{D}. \tag{6. 4}$$

Remark. We may choose for $\mathfrak{T}^2$ the domain

$$\mathfrak{T}^2 = \mathfrak{H}^2[1 - 2\kappa(t)], \tag{6. 5}$$

(see Hypothesis 5. 3) where

$$t = \tfrac{1}{2}\eta_2(\delta). \tag{6. 6}$$

Then our interior domain of comparison is determined by one positive parameter, δ.

(ii) *Exterior domain of comparison.* We shall obtain this domain by applying the *PT*

$$z^*_1 = z_1, \; z^*_2 = z_2/(1 - \sigma z_1^{1/\mu}), \qquad (0 < \sigma < 1, \; \mu > 0), \tag{6. 7}$$

to the domain [11]

[10] Concerning the visualization of the domains of two complex variables see [2] and [5]. (Fig. 3 is a "still" of a model of $\mathfrak{Z}$.)

[11] Note that according to our definition the second vertex of

$$\mathfrak{Z}^2(\theta_2(\gamma) + 2\delta, 2\pi + \theta_1(\gamma) - 2\delta, \rho)$$

lies on the *negative* real axis.

$$(6.8)\quad \mathfrak{G} = \sum_{\gamma \in \mathfrak{R}^2} \mathrm{E}[z_2 = \gamma, |z_1| < 1, z_1 \notin \mathrm{E}(x_1 < -\rho, y_1 = 0), \\ z_1 \notin \mathfrak{F}^2(\theta_2(\gamma) + 2\delta, 2\pi + \theta_1(\gamma) - 2\delta, \rho)]$$

where δ is a sufficiently small positive number, $\rho > 0$, and $\mathfrak{R}^2$ is a fixed domain for which we have $\mathfrak{E}^2 \supset \mathfrak{R}^2 \supset \bar{\mathfrak{F}}^2$.

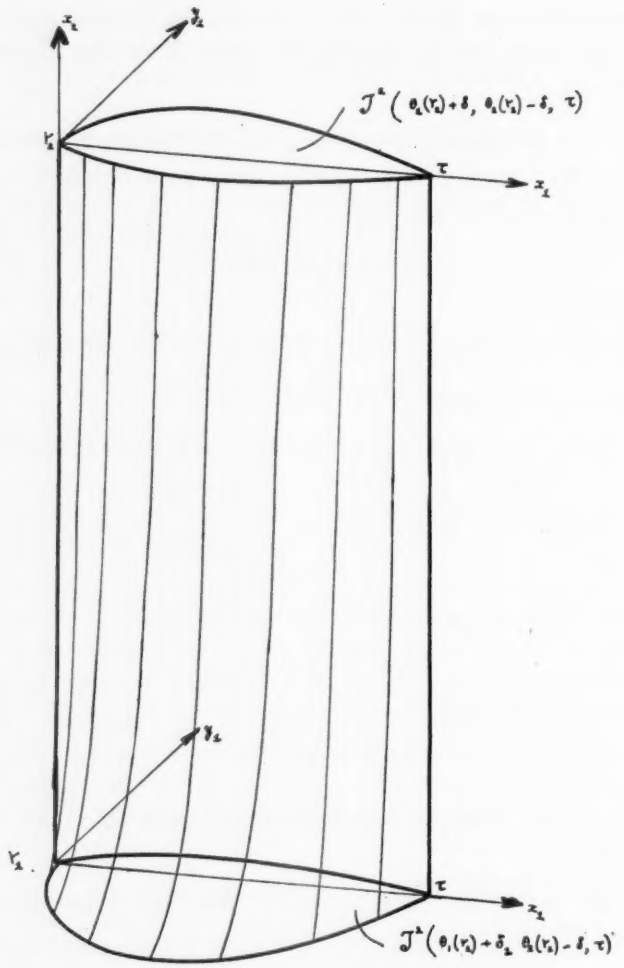

Fig. 3.

An intersection of $\mathfrak{I}$ with space $y_2 = \text{const.}$

Remark. Note that $\mathfrak{G}$ contains only points (z_1, z_2) with $|\arg z_1| < \pi$. The transformation (6.7) is therefore a one-to-one transformation. We may write $z_1^{1/\mu} = (re^{i\phi})^{1/\mu} = r^{1/\mu} e^{i\phi/\mu}$.

Before choosing the values of the parameters in (6.7) and (6.8), we shall prove some very simple lemmas.

LEMMA 3. *For given N, $N > 1$, and t, $0 < t < 1$, and for*

$$\sigma = N/(N+1), \tag{6.9}$$

there exists a $\mu_0 > 0$ such that

$$|\,[1 - \sigma z_1^{1/\mu}]^{-1}\,| > N \tag{6.10}$$

for

$$\mu > \mu_0, \qquad |\,z_1\,| > t. \tag{6.11}$$

Proof. We have

$$|\,1 - \sigma z_1^{1/\mu}\,|^{-2} > |\,1 + \sigma^2 - 2\sigma t^{1/\mu}\cos(\pi/\mu)\,|^{-1} \to (1-\sigma)^{-2} = (N+1)^2 \tag{6.12}$$

as $\mu \to \infty$.

LEMMA 4. *For a fixed σ, $0 < \sigma < 1$, and $\epsilon > 0$, there exists $\mu_1 > 0$ such that for $\mu > \mu_1$,*

$$|\arg z_2 - \arg z^*_2\,| < \epsilon \tag{6.13}$$

[See (6.7)].

Proof. By Hypothesis 5.2a, $|\,z_1\,| < 1/2$. Hence

$$|\arg(1 - \sigma z_1^{1/\mu})\,| \leqq \arctan\,[2^{1/\mu}\sin(\pi/\mu)/(1 - \sigma 2^{-1/\mu})] \to 0, \tag{6.14}$$

as $\mu \to \infty$.

LEMMA 5. *For fixed values of σ $(0 < \sigma < 1)$, δ $(\delta > 0)$, and μ $(\mu > 0)$, there exists a positive χ such that*

$$|\,\theta_k(z^*_2) - \theta_k(z_2)\,| < \delta/2 \tag{6.15}$$

for $|\,z_1\,| < \chi$.

Proof. By (6.7), (5.9) and hypothesis 5.2a we have

$$|\,\theta_k[z_2(1 - \sigma z_1^{1/\mu})] - \theta_k(z_2)\,| \leqq C\chi^{1/\mu}\sigma. \tag{6.16}$$

We shall now determine the parameters in (6.7) and (6.8). We choose a sufficiently small positive δ and write

$$t = (1/2)\eta_2(\delta), \tag{6.17}$$

$$\mathfrak{R}^2 = \mathfrak{H}^2[1 + 2\kappa(t)], \tag{6.18}$$

$$(1/N) = \min_{0 \leqq \phi \leqq 2\pi} R(\phi), \tag{6.19}$$

$$\sigma = N/(N+1). \tag{6.20}$$

See Lemma 2, (5.1) and Hypothesis 5.3.

Next we determine a positive ϵ which satisfies the conditions expressed in Lemma 1 with $k = \kappa(t)$. With these values of t, N, σ and ϵ we find numbers

μ_0 and μ_1 which satisfy the inequalities indicated in Lemmas 3 and 4. Then we choose

(6.21) $$\mu = \max(\mu_0, \mu_1).$$

With this value of μ we determine χ satisfying the conditions of Lemma 5, and such that

(6.22) $$0 < \chi < t.$$

Finally we write

(6.23) $$\rho = \chi \sin \alpha.$$

See Hypothesis 4.2.

With the parameters determined above we construct the domain $\mathfrak{G}$ [see (6.8)] and apply to it the *PT* (6.7). We obtain a domain $\mathfrak{A}$ for which we shall prove that

(6.24) $$\mathfrak{A} \supset \mathfrak{D}.$$

Let (z_1^0, z_2^0) be a point of $\mathfrak{D}$. By (ζ_1^0, ζ_2^0) we denote the point which we obtain from (z_1^0, z_2^0) by the transformation inverse to (6.7), i.e. let

(6.25) $$\zeta_1^0 = z_1^0, \quad \zeta_2^0 = z_1^0[1 - \sigma(z_1^0)^{1/\mu}].$$

We must show that $(z_1^0, z_2^0) \in \mathfrak{A}$. To this end we consider three cases.

1°. $|z_1^0| > t$. By (6.22) and (6.23), $\rho < t$. By (6.8) and (6.18), the intersection of $\mathfrak{G}$ with the plane $z_1 = z_1^0$ is $E[z_1 = z_1^0, z_1 \in \mathfrak{H}^2(1 + 2\kappa(t))]$. Hence it contains the circle $E[z_1 = z_1^0, |z_2| < \min_{0 \leqq \phi \leqq 2\pi} R(\phi)]$. By the transformation (6.7) this circle has been expanded in a ratio greater then $[\min_{0 \leqq \phi \leqq 2\pi} R(\phi)]^{-1}$ [see Lemma 3 and (6.19)]. Thus the intersection of $\mathfrak{A}$ with the plane $z_1 = z_1^0$ contains the whole circle $E[z_1 = z_1^0, |z_2| < 1]$. Since $|z_2^0| < 1$, therefore (z_1^0, z_2^0) is a point of $\mathfrak{A}$.

2°. $\chi \leqq |z_1^0| \leqq t$. By Hypothesis 5.3, z_2^0 is an interior point of $\mathfrak{H}^2[1 + \kappa(t)]$. We obtain the intersection of $\mathfrak{A}$ with $E[z_1 = z_1^0]$ by applying to $\mathfrak{H}^2[1 + 2\kappa(t)]$ the transformation $z^*_2 = z_2 e^{-i\phi}/p$, $pe^{i\phi} = 1 - \sigma(z_1^0)^{1/\mu}$. By Lemma 1 and Lemma 4, the rotation $\zeta = e^{-i\phi} z_2$ applied to the domain $\mathfrak{H}^2[1 + 2\kappa(t)]$ furnishes a domain which still contains $\mathfrak{H}^2[1 + \kappa(t)]$. Since $p < 1$ the transformation $z^*_2 = \zeta/p$ is a positive dilatation. Thus $\mathfrak{A} \cdot E(z_1 = z_1^0)$ contains the domain $\mathfrak{H}^2[1 + \kappa(t)]$. Thus $(z_1^0, z_2^0) \in \mathfrak{A}$.

3°. $|z_1^0| < \chi$. Again we have $z_2^0 \in \mathfrak{H}^2[1 + \kappa(t)]$, and we can show in the same way as in case 2° that $\zeta_2^0 \in \mathfrak{H}^2[1 + 2\kappa(t)]$. In order to show that $(\zeta_1^0, \zeta_2^0) \in \mathfrak{G}$, it will suffice to prove that

(6.26) $$\theta_1(\zeta_2^0) - 2\delta < \arg \zeta_1^0 < \theta_2(\zeta_2^0) + 2\delta.$$

Since $(z_1^0, z_2^0) \in \mathfrak{D}$ and $z_1^0 = \zeta_1^0$, we have by Lemma 2 (A)

$$\theta_1(z_2^0) - \delta < \arg \zeta_1^0 < \theta_2(z_2^0) + \delta, \tag{6.27}$$

since $\chi <$ (6.26) follows immediately by Lemma 5. But $(\zeta_1^0, \zeta_2^0) \in \mathfrak{G}$, therefore $(z_1^{(\nu)}, z_2^{(\nu)}) \in \mathfrak{A}$. The assertion (6.2) is thus proved.

We have constructed an exterior domain of comparison $\mathfrak{A}$. Note that it is determined by one positive parameter, δ.

7. Limit relation for L_2-points. In this section we shall complete the proof of Theorem II. Let $\{(z_1^{(\nu)}, z_2^{(\nu)})\}$, $(\nu = 1, 2, \cdots)$, be a sequence of points which approach the boundary point $Q = (0, 0)$ in the A-sense (see p. 686) as $\nu \to \infty$. We must prove that the expression

$$| z_1^{(\nu)} |^2 K_{\mathfrak{D}}(z_1^{(\nu)}, z_2^{(\nu)}; \bar{z}_1^{(\nu)}, \bar{z}_2^{(\nu)}) \tag{7.1}$$

possesses a non-vanishing limit for $\nu \to \infty$. For the sake of simplicity we shall denote this expression by $K_\nu(\mathfrak{D})$. By $K_\nu(\mathfrak{B})$ we shall denote the analogous expression for some other domain $\mathfrak{B}$. We choose a sufficiently small positive δ and construct the corresponding interior and exterior domains of comparison $\mathfrak{J}$ and $\mathfrak{A}$. For sufficiently large ν, the points $(z_1^{(\nu)}, z_2^{(\nu)})$ lie within $\mathfrak{J}$. Since

$$\mathfrak{J} \subset \mathfrak{D} \subset \mathfrak{A} \tag{7.2}$$

we have by Theorem B

$$K_\nu(\mathfrak{A}) \leqq K_\nu(\mathfrak{D}) \leqq K_\nu(\mathfrak{J}). \tag{7.3}$$

$\mathfrak{J}$ is a standard domain, and therefore

$$\lim_{\nu \to \infty} K_\nu(\mathfrak{J}) = L_{\mathfrak{J}}(0, \phi)$$

exists and is a positive continuous function of ϕ, $\phi = \lim_{\nu \to \infty} \arg z_1^{(\nu)}$. Thus

$$\underline{\lim}\, K_\nu(\mathfrak{A}) \leqq \underline{\lim}\, K_\nu(\mathfrak{D}) \leqq \overline{\lim}\, K_\nu(\mathfrak{D}) \leqq \overline{\lim}\, K_\nu(\mathfrak{J}). \tag{7.4}$$

$\mathfrak{A}$ has been obtained from $\mathfrak{G}$ by the transformation (6.7). The Jacobian of this transformation converges uniformly to 1 as $(z_1, z_2) \to (0, 0)$. By Theorem A (see p. 680)

$$\underline{\lim}\, K_\nu(\mathfrak{A}) = \underline{\lim}\, | z^*{}_1^{(\nu)} |^2 K_{\mathfrak{G}}(z^*{}_1^{(\nu)}, z^*{}_2^{(\nu)}; \bar{z}^*{}_1^{(\nu)}, z^*{}_2^{(\nu)}). \tag{7.5}$$

Consider instead of $\mathfrak{G}$ the domain

$$\mathfrak{M} = \sum_{\gamma \in \mathfrak{N}^2} E[z_2 = \gamma, z_1 \in \mathfrak{F}^2(\theta_2(\gamma) + 2\delta, \pi + \theta_1(\gamma) - 2\delta, \rho')], \tag{7.6}$$

where δ and $\mathfrak{N}^2$ have the previous significance (see p. 695) and

$$\rho' = \rho \sin \alpha. \tag{7.7}$$

Since $\mathfrak{M} \supset \mathfrak{G}$

$$(7.8)\quad |z_1^{(\nu)}|^2 K_{\mathfrak{G}}(z_1^{(\nu)}, z_2^{(\nu)}; \bar{z}_1^{(\nu)}, \bar{z}_2^{(\nu)}) \geqq |z_1^{(\nu)}|^2 K_{\mathfrak{M}}(z_1^{(\nu)}, z_2^{(\nu)}; \bar{z}_1^{(\nu)}, \bar{z}_2^{(\nu)})$$

$\mathfrak{M}$ is a standard domain. Therefore the limit of the right hand side of (7.8) exists. Since $\arg z_1$ is unchanged by the transformation (6.7), this limit is equal to $L_{\mathfrak{M}}(0, \phi)$. Thus we have

$$(7.9)\quad L_{\mathfrak{M}}(0, \phi) \leqq \underline{\lim}\, K_\nu(\mathfrak{D}).$$

The *PT*

$$(7.10)\quad \zeta_1 = z_1/[1 + z_1/\rho'],\ \zeta_2 = z_2$$

maps $\mathfrak{M}$ into an $\mathfrak{S}$-domain

$$(7.11)\quad \mathfrak{S}_a = \mathop{\mathrm{S}}_{\gamma \in \mathfrak{T}^2} \mathrm{E}[z_2 = \gamma,\ z_1 \in \mathfrak{S}^2(\theta_1(\gamma) - 2\delta,\ \theta_2(\gamma) + 2\delta)]$$

The *PT*

$$(7.12)\quad \zeta_1 = z_1/[1 - \epsilon^{-1} z_1],\ \zeta_2 = z_2$$

maps $\mathfrak{J}$ into an $\mathfrak{S}$-domain

$$\mathfrak{S}_i = \mathop{\mathrm{S}}_{\gamma \in \mathfrak{T}^2} \mathrm{E}[z_2 = \gamma,\ z_1 \in \mathfrak{S}^2(\theta_1(\gamma) + \delta,\ \theta_2(\gamma) - \delta)].$$

The Jacobians of the transformations (7.10) and (7.12) are both equal to 1 at $(0, 0)$, and in both cases we have

$$(7.13)\quad |\arg z_1 - \arg \zeta_1| \to 0,\quad \zeta_1 \to 0,\ z_1 \to 0.$$

By Theorem A,

$$(7.14)\quad L_{\mathfrak{M}}(0, \phi) = L_{\mathfrak{S}_a}(0, \phi),\quad L_{\mathfrak{J}}(0, \phi) = L_{\mathfrak{S}_i}(0, \phi).$$

By (7.4), (7.5), (7.9) and (7.14) we finally have

$$(7.15)\quad L_{\mathfrak{S}_a}(0, \phi) \leqq \underline{\lim}\, K_\nu(\mathfrak{D}) \leqq \overline{\lim}\, K_\nu(\mathfrak{D}) \leqq L_{\mathfrak{S}_i}(0, \phi).$$

Now we apply to $\mathfrak{S}_i$ the *PT*

$$(7.16)\quad \zeta_1 = z_1^{1+b},\quad \zeta_2 = (1 + 2\kappa)(1 - 2\kappa)^{-1} z_2,$$

where

$$(7.17)\quad \kappa = \kappa(t)$$

and

$$(7.18)\quad b = [3\delta + 4C\kappa^2(t)]/[\alpha - 2\delta].$$

(We have then

$$(7.19)\quad \arg \zeta_1 = (1 + b) \arg z_1.)$$

We denote by $\mathfrak{S}_j$ the domain obtained in this manner, and state that

$$(7.20)\quad \mathfrak{S}_j \supset \mathfrak{S}_a.$$

In order to prove this assertion we must only show that

$$(7.21)\quad |(1+b)[\theta_k\left(\frac{1-2\kappa}{1+2\kappa}z_2\right)-(-1)^k\delta]| > |\theta_k(z_2)+2(-1)^k\delta|.$$

(For, the transformation $\zeta=\dfrac{1+2\kappa}{1-2\kappa}z$ maps $\mathfrak{T}^2$ into $\mathfrak{R}^2$.) We have, say for $k=2$, by the mean value theorem,

$$(7.22)\qquad |\theta_2\left(\frac{1-2\kappa}{1+2\kappa}z_2\right)-\theta_2(z_2)| \leqq 4C\kappa^2,$$

(see (5.9)) and therefore

$$\begin{aligned}(7.23)\quad &(1+b)[\theta_2\left(\frac{1-2\kappa}{1+2\kappa}z_2\right)-\delta]-\theta_2(z_2)-2\delta\\ &\geqq -4C\kappa^2+b\alpha-b\delta-3\delta\\ &= b(\alpha-\delta)-(3\delta+4C\kappa^2) > b(\alpha-2\delta)-(3\delta+4C\kappa^2)=0.\end{aligned}$$

Thus (7.20) is proved. By Theorem B and (7.5),

$$(7.24)\qquad L_{\mathfrak{S}_j}(0,\phi) \leqq L_{\mathfrak{S}_a}(0,\phi) \leqq L_{\mathfrak{S}_i}(0,\phi).$$

Since

$$(7.25)\qquad \frac{\partial(\zeta_1,\zeta_2)}{\partial(z_1,z_2)} = \frac{1+2\kappa}{1-2\kappa}(1+b)z_1{}^b,$$

we have by Theorem A,

$$\begin{aligned}(7.26)\quad &\mathbf{K}_{\mathfrak{S}_j}(\zeta_1,\zeta_2;\bar\xi_1,\bar\xi_2)\\ &\qquad = (1+2\kappa)^2(1-2\kappa)^{-2}(1+b)^{-2}\,|\,\zeta_1\,|^{-2b}\mathbf{K}_{\mathfrak{S}_i}(z_1,z_2;\bar z_1,\bar z_2),\end{aligned}$$

and

$$\begin{aligned}(7.27)\quad &|\,z_1\,|^{2(1+b)}\mathbf{K}_{\mathfrak{S}_j}(\zeta_1,\zeta_2;\bar\xi_1,\bar\xi_2)\\ &\qquad = (1-2\kappa)^2(1-2\kappa)^{-2}(1+b)^{-2}\,|\,z_1\,|^2\mathbf{K}_{\mathfrak{S}_i}(z_1,z_2;\bar z_1,\bar z_2),\end{aligned}$$

i. e.

$$(7.28)\qquad |\,\zeta_1\,|^2\mathbf{K}_{\mathfrak{S}_j}(\zeta_1,\zeta_2;\bar\zeta_1,\bar\zeta_2) = |\,z_1\,|^2\mathbf{K}_{\mathfrak{S}_i}(z_1,z_2;\bar z_1,\bar z_2).$$

Hence it follows by (7.19) that

$$(7.29)\quad L_{\mathfrak{S}_j}(0,\phi) = (1-2\kappa)^2(1+2\kappa)^{-2}(1+b)^{-2}L_{\mathfrak{S}_i}\,[0,\phi/(1+b)].$$

Since $L_{\mathfrak{S}_i}(0,\phi)$ is a continuous function of ϕ, and since for sufficiently small δ, b and κ will be arbitrarily small [see (7.18), (7.17), (5.15a) and (5.4)], the difference

$$(7.30)\qquad |\,L_{\mathfrak{S}_j}(0,\phi)-L_{\mathfrak{S}_i}(0,\phi)|$$

can be made arbitrarily small. The same is, therefore, true for

$$(7.31)\qquad |\,\underline{\lim}\,\mathbf{K}_\nu(\mathfrak{D})-\overline{\lim}\,\mathbf{K}_\nu(\mathfrak{D})|$$

[see (7.15) and (7.24)]. Thus the limit

$$\lim K_\nu(\mathfrak{D}) = L_{\mathfrak{D}}(0, \phi) \tag{7.32}$$

exists, and Theorem II is proved.

By using analogous methods we can formulate sufficient conditions in order that all boundary points $(0, Z_2)$ situated in a sufficiently small neighborhood of Q be L-points of the second order. To each of these points we then associate the limit $L_{\mathfrak{D}}(Z_2, \phi)$, which we define for $(0, Z_2)$ in the similar way as before $L_{\mathfrak{D}}(0, \phi)$ for $(0, 0)$. It is then possible to prove that $L_{\mathfrak{D}}(Z_2, \phi)$ is a continuous function of Z_2 and ϕ.

I take the opportunity of thanking Lipman Bers for some suggestions in connection with this paper.

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Brown University.

PLANE SECTIONS THROUGH A POINT OF A NON-HOLONOMIC SURFACE.*

By BUCHIN SU.

In a previous paper [1] we showed that if there are given two arbitrary non-asymptotic tangents t_1 and t_2 of a (holonomic) surface S at an ordinary point O and a plane π_1 through t_1, we can always determine a quadric Q through the two asymptotic tangents of S at O such that the osculating conic of the section of S produced by π_1 and that of the section produced by the corresponding plane π_2 through t_2 lie on Q. This quadric has at O a contact of the second order with S and forms a pencil when π_1 turns about t_1. The present paper develops a similar theory for a non-holonomic surface in ordinary space. According to the non-holonomy we obtain some results different from those in the holonomic case. Take, for example, a quadric Q of the type referred to; there exist in general *four* other tangents through each of which a plane can be drawn such that the osculating conic of the corresponding section at O lies on Q. Moreover, we can determine certain lines and quadratic cones at a point of a non-holonomic surface by means of the neighborhoods of the third and the fourth order, which do not exist in the holonomic case. Especially, a covariant line is constructed by means of the neighborhood of the third order and a new extension of the Darboux pencil of quadrics is established.

1. Let a non-holonomic surface V_3^2 in ordinary space be defined by an equation of Pfaff, namely,

$$dz = p(x, y, z)\,dx + q(x, y, z)\,dy, \tag{1}$$

where the functions p and q do not satisfy the condition of integrability. At a generic point O of V_3^2 we can determine a projective system of reference in which the two asymptotic tangents, supposed to be distinct at O, are $x = z = 0$ and $y = z = 0$, so that the tangent plane of V_3^2 at O is $z = 0$, and such that the axis z and "the line at infinity" of the tangent plane are in correspondence of the projectivity of cell due to Bompiani,[2] so that at 0

* Received June 1, 1942.

[1] Cf. B. Su, "Osculating conics of the plane sections through a point of a surface," *American Journal of Mathematics*, vol. 65 (1943), pp. 439-449.

[2] Cf. E. Bompiani, "Sulle varietà anolonome, I. Alcuni teoremi generali; II. Le V_3^2 di S_5 proiettivo," *Rendiconti dei Lincei*, (6), vol. 27 (1938), pp. 37-52.

$$p = q = 0, \quad p_x = q_y = 0, \quad p_z = q_z = 0, \tag{2}$$

where the subscript denotes partial differentiation with respect to the indicated argument.

Consider the section of V_3^2 produced by a plane π through a non-asymptotic tangent t:

$$z = y - nx = 0; \tag{3}$$

if π be expressed by the equation

$$z = \rho(y - nx), \tag{4}$$

where $\rho \neq 0, \neq \infty$, then the osculating conic of the section at O is given by the equations [3]

$$\left\{\begin{aligned} &\frac{n}{2\rho}(p_y + q_x)Y = \frac{n^2}{4\rho^2}(p_y + q_x)^2x^2 - \tfrac{1}{6}\left(\frac{A_3}{\rho} + \frac{A_0 n}{\rho^2}\right)xY \\ &+ \left\{\frac{1}{qn^2(p_y + q_x)^2}\left(A_3 + \frac{A_0 n}{\rho}\right)^2 - \frac{1}{12n(p_y + q_x)}\left(B_4 + \frac{C_3}{\rho} + \frac{D_0 n}{\rho^2}\right)\right\}Y^2 = 0, \\ &z = \rho Y, \end{aligned}\right. \tag{5}$$

where we have placed

$$Y = y - nx \tag{6}$$

and

$$\left\{\begin{aligned} A_0 &= (p_y + q_x)(p_y + 2q_x), \\ A_3 &= p_{xx} + (2p_{xy} + q_{xx})n + (2q_{xy} + p_{yy})n^2 + q_{yy}n^3, \\ D_0 &= (p_y + q_x)(p_y + 2q_x)(p_y + 3q_x), \\ C_3 &= q_{yy}(7p_y + 9q_x)n^3 + \{2p_{yy}(2p_y + 3q_x) + q_{xy}(11p_y + 15q_x)\}n^2 \\ &\quad + \{p_{xy}(5p_y + 9q_x) + 2q_{xx}(2p_y + 3q_x)\}n + p_{xx}(3q_x + p_y), \\ B_4 &= p_{xxx} + \{3p_{xxy} + 3p_{xz}(p_y + q_x) + q_{xxx}\}n \\ &\quad + \{3p_{xyy} + 3p_{yz}(p_y + q_x) + 3q_{xxy} + 3q_{xz}(p_y + q_x)\}n^2 \\ &\quad + \{p_{yyy} + 3q_{xyy} + 3q_{yz}(p_y + q_x)\}n^3 + q_{yyy}n^4. \end{aligned}\right. \tag{7}$$

2. Let us now take a tangent quadric Q of V_3^2 at O and determine whether three or more osculating conics of V_3^2 lie on Q. Since Q is given by an equation of the form

$$z = ax^2 + 2bxy + cy^2 + z(k_2x + k_3y + k_4z), \tag{8}$$

---

[3] Cf. B. Su, "On the projective differential geometry of a non-holonomic surface in ordinary space," *Annali di Matematica*, (4), vol. 19 (1940), pp. 289-313. The author is much indebted to Prof. Enea Bortolotti for his valuable correction and supplement. Cf. E. Bortolotti, "Sulla geometria proiettiva differenziale di una superficie anolonoma nello spazio ordinario," *Ibidem*, vol. 19 (1940), pp. 315-325.

a, b, c, k_2, k_3, k_4 being constants, the section of Q produced by the plane π is the conic

$$(9)\qquad \begin{cases} -\frac{1}{\rho}Y+\frac{1}{\rho^2}(a+2bn+cn^2)x^2+\left\{2\,\frac{b+cn}{\rho^2}+\frac{k_2+k_3n}{\rho}\right\}xY \\ \qquad\qquad +\left(k_4+\frac{k_3}{\rho}+\frac{c}{\rho^2}\right)Y^2=0, \\ z=\rho Y. \end{cases}$$

A comparison of these equations with (5) shows that the latter conic becomes an osculating conic of V_3^2 at O when and only when

$$(10)\qquad \begin{cases} a+2bn+cn^2=\frac{1}{2}(p_y+q_x)n, \\ \frac{n}{2}(p_y+q_x)\{2\,\frac{b+cn}{\rho}+k_2+k_3n\}=\frac{1}{6}\left(\frac{A_0n}{\rho}+A_3\right), \\ -\frac{n}{2}(p_y+q_x)\left(k_4+\frac{k_3}{\rho}+\frac{c}{\rho^2}\right) \\ =\frac{1}{qn^2(p_y+q_x)^2}\left(A_3+\frac{A_0n}{\rho}\right)^2-\frac{1}{12n(p_y+q_x)}\left(B_4+\frac{C_3}{\rho}+\frac{D_0n}{\rho^2}\right). \end{cases}$$

For a quadric Q given by (8) there are in general two tangents (3) satisfying the first equation of (10), but no value of ρ can be obtained so as to satisfy the remaining two equations. Therefore we may conclude that no osculating conic of the section of V_3^2 at O can lie on a general tangent quadric Q of V_3^2 at O.[4]

There is, however, an exceptional case where the first equation of (10) is satisfied identically in n, so that

$$a=0,\quad b=\tfrac{1}{4}(p_y+q_x),\quad c=0.$$

The corresponding quadric (8) becomes

$$(11)\qquad z=\tfrac{1}{2}(p_y+q_x)xy+z(k_2x+k_3y+k_4z),$$

which has at O a contact of the second order with V_3^2; while the conditions (10) take the form

$$(12)\qquad k_2+k_3n=-\tfrac{1}{6}(p_y-q_x)\,\frac{1}{\rho}+\frac{A_3}{3n(p_y+q_x)},$$

$$(13)\qquad k_4+\frac{k_3}{\rho}=\frac{3n(p_y+q_x)B_4-4A_3^2}{18n^3(p_y+q_x)^3}+\frac{3C_3-8(p_y+2q_x)A_3}{18n^2(p_y+q_x)^2\rho}-\frac{(p_y+2q_x)(p_y-q_x)}{18n(p_y+q_x)\rho^2}.$$

[4] If we determine ρ from the second equation of (10), where n is a root of the first equation, and substitute it into the last equation, there is found a relation between a, b, c, k_2, k_3 and k_4, which does not hold in the general case.

3. As an analogue of the problem for a surface we propose to determine a quadric Q of the system furnished by equation (11) containing both the osculating conics of sections of V_3^2 produced by planes through two arbitrary non-asymptotic tangents t_1, t_2 at O. Write the equations of t_i $(i=1,2)$ as

$$z = y - n_i x = 0$$

and the equation of the corresponding plane π_i as

$$z = \rho_i (y - n_i x);$$

it follows from (12) and (13) that four conditions

$$(14)\quad k_2 + k_3 n_i = -\tfrac{1}{6}(p_y - q_x)\frac{1}{\rho_i} + \frac{A_3(n_i)}{3n_i(p_y+q_x)},$$

$$(15)\quad k_4 + \frac{k_3}{\rho_i} = \frac{3n_i(p_y+q_x)B_4(n_i) - 4A_3^2(n_i)}{18n_i^3(p_y+q_x)^3} + \frac{3C_3(n_i) - 8(p_y+2q_x)A_3(n_i)}{18n_i^2(p_y+q_x)^2\rho_i} - \frac{(p_y+2q_x)(p_y-q_x)}{18n_i(p_y+q_x)\rho_i^2} \quad (i=1,2)$$

should be satisfied for arbitrarily assigned n_1, n_2, where $A_3(n_i)$, $C_3(n_i)$ and $B_4(n_i)$ are given by (7), n being replaced by n_i.

Solving (14) for k_2 and k_3 we have

$$(16)\quad \begin{cases} k_2 = -\tfrac{1}{6}(p_y - q_x)\dfrac{\rho_1 n_1 - \rho_2 n_2}{\rho_1\rho_2(n_1 - n_2)} \\ \qquad + \dfrac{1}{3(p_y+q_x)}\left\{\dfrac{n_1+n_2}{n_1 n_2}p_{xx} + 2p_{xy} + q_{xx} - q_{yy}n_1 n_2\right\} \\ k_3 = \tfrac{1}{6}(p_y - q_x)\dfrac{\rho_1 - \rho_2}{\rho_1\rho_2(n_1 - n_2)} \\ \qquad + \dfrac{1}{3(p_y+q_x)}\left\{-\dfrac{1}{n_1 n_2}p_{xx} + 2q_{xy} + p_{yy} + q_{yy}(n_1+n_2)\right\}. \end{cases}$$

In order that the two equations (15) should be compatible, it is necessary and sufficient that

$$k_3\left(\frac{1}{\rho_1} - \frac{1}{\rho_2}\right) = \frac{1}{18(p_y+q_x)^3}\left[3(p_y+q_x)\left\{\frac{1}{n_1^2}B_4(n_1) - \frac{1}{n_2^2}B_4(n_2)\right\} - 4\left\{\frac{1}{n_1^3}A_3^2(n_1) - \frac{1}{n_2^3}A_3^2(n_2)\right\}\right]$$
$$+ \frac{1}{18(p_y+q_x)^2}\left[3\left\{\frac{1}{n_1^2\rho_1}C_3(n_1) - \frac{1}{n_2^2\rho_2}C_3(n_2)\right\} - 8(p_y+2q_x)\left\{\frac{1}{n_1^2\rho_1}A_3(n_1) - \frac{1}{n_2^2\rho_2}A_3(n_2)\right\}\right]$$
$$- \frac{(p_y+2q_x)(p_y-q_x)}{18(p_y+q_x)}\left\{\frac{1}{\rho_1^2 n_1} - \frac{1}{\rho_2^2 n_2}\right\}.$$

Substitution of k_3 given by (16) shows that the two planes π_1 and π_2 are in the following correspondence:

$$
\begin{aligned}
(17)\qquad & \frac{p_y - q_x}{n_1 - n_2}\left\{\frac{(2n_1+n_2)p_y+(n_1+2n_2)q_x}{n_1\rho_1^2} - 6(p_y+q_x)\,\frac{1}{\rho_1\rho_2}\right. \\
& \qquad \left. + \frac{(n_1+2n_2)p_y+(2n_1+n_2)q_x}{n_2\rho_2^2}\right\} \\
& + \frac{1}{p_y+q_x}\left[6(p_y+q_x)\left\{\frac{p_{xx}}{n_1n_2} - (2q_{xy}+p_{yy}) - q_{yy}(n_1+n_2)\right\}\right. \\
& \qquad \left. + 3\,\frac{1}{n_1^2}\,C_3(n_1) - 8(p_x+2q_y)\,\frac{1}{n_1^2}\,A_3(n_1)\right]\frac{1}{\rho_1} \\
& - \frac{1}{p_y+q_x}\left[6(p_y+q_x)\left\{\frac{p_{xx}}{n_1n_2} - (2q_{xy}+p_{yy}) - q_{yy}(n_1+n_2)\right\}\right. \\
& \qquad \left. + 3\,\frac{1}{n_2^2}\,C_3(n_2) - 8(p_y+2q_x)\,\frac{1}{n_2^2}\,A_3(n_2)\frac{1}{\rho_2}\right. \\
& + \frac{1}{(p_y+q_x)^2}\left[3(p_y+q_x)\left\{\frac{B_4(n_1)}{n_1^2} - \frac{B_4(n_2)}{n_2^2}\right\}\right. \\
& \qquad \left. - 4\left\{\frac{A_3^2(n_1)}{n_1^3} - \frac{A_3^2(n_2)}{n_2^3}\right\}\right] = 0.
\end{aligned}
$$

Thus we have the

THEOREM. *If two planes through a generic point O of a non-holonomic surface V_3^2 are drawn such that each of them passes through one of two arbitrarily assigned non-asymptotic tangents and the osculating conics of the sections of V_3^2 produced by these planes lie on a quadric Q tangent to V_3^2 at O, then Q must have at O a contact of the second order with V_3^2 and the correspondence between these two planes is* $(2,2)$.

In the holonomic case (17) becomes a linear equation in $1/\rho_1$ and $1/\rho_2$, and therefore the correspondence is a perspectivity, as we have shown before.[5]

4. It is natural to inquire whether other osculating conics of V_3^2 at O belong to the same quadric Q thus determined. If there is such a conic, then, expressing the plane of the conic by the equation

$$(18)\qquad z = \rho_3(y - n_3x),$$

we have

$$(19)\qquad k_2 + k_3n_3 = -\tfrac{1}{6}(p_y - q_x)\,\frac{1}{\rho_3} + \frac{A_3(n_3)}{3n_3(p_y+q_x)},$$

[5] Cf. B. Su, *loc. cit.*[1]

$$(20)\quad k_4+\frac{k_3}{\rho_3}=\frac{3n_3(p_y+q_x)B_4(n_3)-4A_3^2(n_3)}{18n_3^3(p_y+q_x)^3}$$
$$+\frac{3C_3(n_3)-8(p_y+2q_x)A_3(n_3)}{18n_3^2(p_y+q_x)^2\rho_3}-\frac{(p_y+2q_x)(p_y-q_x)}{18n_3(p_y+q_x)\rho_3^2}.$$

Substituting the value of k_3 given by (16) into (19) and reducing, we obtain

$$(21)\quad (p_y-q_x)\left\{(n_2-n_3)\frac{1}{\rho_1}+(n_3-n_1)\frac{1}{\rho_2}+(n_1-n_2)\frac{1}{\rho_3}\right\}$$
$$+\frac{2}{p_y+q_x}(n_1-n_2)(n_2-n_3)(n_3-n_1)\{p_{xx}(n_1n_2n_3)^{-1}+q_{yy}\}=0.$$

As the coefficient of the term ρ_3^{-1} in the last equation is different from zero, we can express ρ_3^{-1} as a function of n_3. Further substitution of the latter into (20) suffices to demonstrate that the resulting equation is an algebraic equation of the sixth degree in n_3.[6]

For each root of this equation we can find the corresponding ρ_3 from (20) and therefore obtain the osculating conic on the quadric Q in consideration. Hence we obtain the

THEOREM. *If a quadric Q contains two osculating conics of V_3^2 at O with distinct tangents, then it contains also four other osculating conics of V_3^2 at O.*

For a holonomic surface the equation (21) becomes particularly simple and *uniquely* determines the third tangent.[7]

Conversely, *on a quadric which has at O a contact of the second order with V_3^2 there are in general six osculating conics of V_3^2 at O.*

In fact, the equation of such a quadric is of the form (11), where k_2, k_3, k_4 are given *a priori*. If ρ be eliminated from (12) and (13), the resulting equation is evidently of the sixth degree in n and consequently gives six tangents of V_3^2 at O. In virtue of (12) we have the corresponding value of ρ and therefore the osculating conic of V_3^2 at O.

5. On account of the correspondence (17) between the planes π_1 and π_2 we can further find a line L_{12} associated with the given tangents t_1 and t_2. For this purpose, let us consider the two planes π_2' and π_2'' corresponding to

---

[6] Two roots of this equation are evidently n_1 and n_2; to these correspond the original conics.

[7] Cf. Su, *loc. cit.*[1]

a plane π_1 under (17) and construct the harmonic conjugate $\bar{\pi}_2$ of the tangent plane $z=0$ with respect to π_2' and π_2''; the equation to $\bar{\pi}_2$ is obviously of the form

$$\tfrac{1}{2}(1/\rho_2' + 1/\rho_2'')z = y - n_2 x. \tag{22}$$

where $1/\rho_2'$, $1/\rho_2''$ are roots of the quadratic equation (17) in $1/\rho_2$. A simple calculation shows that these planes π_1 and $\bar{\pi}_2$ are in perspective position, the line of intersection being on the same plane π_{12}:

$$\begin{aligned}(23)\quad & 2(p_y^2 - q_x^2)[(p_y + 2q_x)y + (2p_y + q_x)n_2 x] \\ & - z[p_{xx}\{6(p_y + q_x)(1/n_1 - 1/n_2) + (p_y - q_x)(1/n_2)\} - (p_y + 5q_x)p_{xy} \\ & + 2(2p_y + q_x)q_{xx} - 6(p_y + q_x)q_{yy}n_2(n_1 - n_2) + (p_y - q_x)q_{yy}n_2^2 \\ & + \{(5p_y + q_x)q_{xy} - 2(p_y + 2q_x)p_{yy}\}n_2] = 0.\end{aligned}$$

In a similar way we may define the plane π_{21} by interchanging π_1 and π_2. From the equation (23) and that obtained by interchanging n_1 and n_2 in (23) it follows that the intersection L_{12} of π_{12} and π_{21} is given by the equations

$$\begin{cases} 2(2p_y + q_x)(p_y^2 - q_x^2)x + z[(7p_y + 5q_x)q_{yy}(n_1 + n_2) \\ \quad + 2(p_y + 2q_x)p_{yy} - (5p_y + q_x)q_{xy} - (11p_y + 13q_x)p_{xx}(n_1 n_2)^{-1}] = 0. \\ 2(2q_x + p_y)(q_x^2 - p_y^2)y + z[(7q_x + 5p_y)p_{xx}(1/n_1 + 1/n_2) \\ \quad + 2(q_x + 2p_y)q_{xx} - (5q_x + p_y)p_{xy} - (11q_x + 13p_y)q_{yy}n_1 n_2] = 0. \end{cases} \tag{24}$$

If we consider all the lines L_{12} corresponding to the tangents apolar with respect to the asymptotic tangents at O, then the locus is a quadratic cone, namely,

$$\begin{aligned}(25)\quad & (11p_y + 13q_x)(11q_x + 13p_y)p_{xx}q_{yy}z^2 \\ & = [2(2p_y + q_x)(p_y^2 - q_x^2)x + \{2(p_y + 2q_x)p_{yy} - (5p_y + q_x)q_{xy}\}z] \\ & \cdot [2(2q_x + p_y)(q_x^2 - p_y^2)y + \{2(q_x + 2p_y)q_{xx} - (5q_x + p_y)p_{xy}\}z].\end{aligned}$$

The polar line $\mathfrak{L}$ of the tangent plane $z=0$ with respect to the cone (25) is determined by the neighborhood of the third order of V_3^2 at O. From (25) we have the equations of $\mathfrak{L}$:

$$\begin{cases} y = \dfrac{2(2p_y + q_x)q_{xx} - (p_y + 5q_x)p_{xy}}{2(p_y + 2q_x)(p_y^2 - q_x^2)}\, z, \\ x = \dfrac{2(2q_x + p_y)p_{yy} - (q_x + 5p_y)q_{xy}}{2(q_x + 2p_y)(q_x^2 - p_y^2)}\, z. \end{cases} \tag{26}$$

6. We are now in a position to investigate the line of intersection of two planes π_1, π_2 in the correspondence (17).

In virtue of (7) and (17) there is no difficulty in determining the locus of the line of intersection. The result of carrying out the computation is as follows:

$$
\begin{aligned}
(27)\qquad & (p_y - q_x)\left\{\frac{p_y+2q_x}{n_1n_2}\,y^2+(2p_y+q_x)x^2\right\}\\
&+\frac{yz}{p_y+q_x}\Big[(5p_y+7q_x)p_{xx}\,\frac{n_1+n_2}{n_1^2n_2^2}+\{(p_y+5q_x)p_{xy}-2(2p_y+q_x)q_{yy}\\
&\qquad\qquad +(13p_y+11q_x)q_{yy}\Big]\\
&-\frac{xz}{p_y+q_x}\Big[(11p_y+13q_x)\,\frac{p_{xx}}{n_1n_2}+(q_x+5p_y)q_{xy}-2(2q_x+p_y)p_{yy}\\
&\qquad\qquad +(7p_y+5q_x)q_{yy}(n_1+n_2)\Big]\\
&+\Big[4\left(\frac{p_{xx}^2}{n_1^3n_2^3}-q_{yy}^2\right)(n_1^2+n_1n_2+n_2^2)\\
&\qquad +8\left\{\frac{p_{xx}(2p_{xy}+q_{xx})}{n_1^2n_2^2}-(2q_{xy}+p_{yy})q_{yy}\right\}(n_1+n_2)\\
&+4\{(2p_{xy}+q_{xx})^2+2p_{xx}(2q_{xy}+p_{yy})\}\,\frac{1}{n_1n_2}\\
&-4\{(2q_{xy}+p_{yy})^2+2q_{yy}(2p_{xy}+q_{xx})\}\\
&+3(p_y+q_x)\left\{-p_{xxx}\,\frac{n_1+n_2}{n_1^2n_2^2}+p_{yyy}+3q_{xyy}\right.\\
&+3q_{yz}(p_y+q_x)+q_{yyy}(n_1+n_2)\\
&\left.-(3p_{xxy}+3p_{xz}(p_y+q_x)+q_{xxx})\,\frac{1}{n_1n_2}\right\}\Big](p_y+q_x)^{-2}z^2=0.
\end{aligned}
$$

This quadratic cone K_{12} associated with two non-asymptotic tangents t_1 and t_2 is evidently determined by the neighborhood of the fourth order of V_3^2 at O. It is easily seen that the polar of the tangent plane of V_3^2 at O with respect to K_{12} just coincides with the line L_{12} associated with the same pair of tangents.

In particular, we consider the three lines L_{12} associated with two of *the Segre tangents*

$$(28)\qquad p_{xx}x^3-q_{yy}y^3=0.$$

Put

$$(29)\qquad \lambda=(p_{xx}/q_{yy})^{1/3},\quad \epsilon^3=1\quad(\epsilon\neq 1);$$

then the lines in question are given by

$$
\begin{aligned}
(30)\qquad & x:y:z\\
&=\frac{p_y+2q_x}{p_y^2-q_x^2}\{4(p_y+2q_x)q_{yy}\lambda\epsilon^r+)5p_y+q_x)q_{xy}-2(p_y+2q_x)p_{yy}\}\\
&:\frac{q_x+2p_y}{q_x^2-p_y^2}\{4(q_x+2p_y)p_{xx}\,\frac{1}{\lambda}\,\epsilon^{2r}+(5q_x+p_y)p_{xy}-2(q_x+2p_y)q_{xx}\}\\
&:2(p_y+2q_x)(q_x+2p_y)\qquad\qquad (r=1,2,3).
\end{aligned}
$$

Consequently, *the harmonic conjugate of the tangent plane $z=0$ with respect to the trihedron formed by these lines is the line $\mathfrak{L}$,* as may easily be derived from (30) and (26).

The same result holds for three tangents in the cubic involution

$$p_{xx}x^3 = \mu q_{yy}y^3 = 0, \tag{31}$$

and especially for *three Darboux tangents.*

7. It is worth noticing that the quadratic cone K_{12} associated with two Darboux tangents or two Segre tangents gives certain covariant lines. Take, for example, two Segre tangents:

$$\begin{cases} z = y - n_i x = 0 & (i = 1, 2), \\ n_1 = \lambda\epsilon^r, \quad n_2 = \lambda\epsilon^s & \\ & (r \neq s,\ s \neq t,\ t \neq r; \quad r, s, t = 1, 2, 3) \end{cases} \tag{32}$$

and substitute them into (27); we easily arrive at the cone:

$$\begin{aligned} &(p_y - q_x)\left\{\frac{p_y + 2q_x}{\lambda^2}\,\epsilon^t y^2 + (2p_y + q_x)x^2\right\} \\ &+ \frac{yz}{p_y + q_x}\left[4(2p_y + q_x)q_{yy} + \frac{1}{\lambda^2}\{(p_y + 5q_x)p_{xy} - 2(2p_y + q_x)q_{xx}\}\epsilon^t\right] \\ &- \frac{xz}{p_y + q_x}\left[4(2q_x + p_y)p_{xx}\frac{\epsilon^t}{\lambda^2} + (5p_y + q_x)q_{xy} - 2(p_y + 2q_x)p_{yy}\right] \\ &+ \frac{z^2}{(p_y + q_x)^2}\left[3(p_y + q_x)\left\{\frac{p_{xxx}}{p_{xx}}q_{yy} + 3q_{xyy} + 3q_{yz}(p_y + q_x) + p_{yyy}\right.\right. \\ &\left. - (q_{xxx} + 3p_{xxy} + 3p_{xz}(p_y + q_x) + \frac{q_{yyy}}{q_{yy}}\,p_{xx})\lambda^{-2}\epsilon^t\right\} \\ &- 4\{(2q_{xy} + p_{yy})^2 + 4(2p_{xy} + q_{xx})q_{yy}\} \\ &\left. + 4\{(2p_{xy} + q_{xx})^2 + 4(2q_{xy} + p_{yy})p_{xx}\}\lambda^{-2}\epsilon^t\right] = 0. \end{aligned} \tag{33}$$

Consequently *the three quadratic cones K_{12} associated with two of three Segre tangents belong to a pencil and therefore determine four lines of intersection,* namely, the lines given by the equations

$$\begin{aligned} &(p_y - q_x)(p_y + 2q_x)y^2 + \frac{yz}{p_y + q_x}\{(p_y + 5q_x)p_{xy} - 2(2p_y + q_x)q_{xx}\} \\ &- 4\,\frac{p_{xx}(p_y + 2q_x)}{p_y + q_x}\,xz + \frac{z^2}{(p_y + q_x)^2}\left[4(2p_{xy} + q_{xx})^2 + 16(2q_{xy} + p_{yy})p_{xx}\right. \\ &\left. - 3(p_y + q_x)\{q_{xxx} + 3p_{xxy} + 3p_{xz}(p_y + q_x) + \frac{q_{yyy}}{q_{yy}}\,p_{xx}\}\right] = 0, \end{aligned} \tag{34}$$

$$(35)\quad (q_x - p_y)(q_x + 2p_y)x^2 + \frac{xz}{p_y + q_x}\{(q_x + 5p_y)q_{xy} - 2(2q_x + p_y)p_{yy}\}$$
$$- 4\frac{q_{yy}(q_x + 2p_y)}{p_y + q_x}\, yz + \frac{z^2}{(p_y + q_x)^2}[4(2q_{xy} + p_{yy})^2 + 16(2p_{xy} + q_{xx})q_{yy}$$
$$- 3(p_y + q_x)\{p_{yyy} + 3q_{xyy} + 3q_{yz}(p_y + q_x) + \frac{p_{xxx}}{p_{xx}}\, q_{yy}\}] = 0.$$

The equations (34) and (35) represent the cones in the pencil touching the tangent plane $z = 0$ along the asymptotic tangents $y = z = 0$ and $x = z = 0$ respectively. The polar plane of the tangent $x = z = 0$ with respect to (34) and that of $y = z = 0$ with respect to (35) intersect in the line $\mathfrak{L}$.

Moreover, a covariant cone similar to (25) is found as the locus of the polar of $z = 0$ with respect to a cone in the pencil above quoted, namely,

$$(36)\quad 16p_{xx}q_{yy}(p_y + 2q_x)(q_x + 2p_y)z^2$$
$$= [2(p_y^2 - q_x^2)(2p_y + q_x)x + \{2(p_y + 2q_x)p_{yy} - (q_x + 5p_y)q_{xy}\}z]$$
$$\cdot [2(q_x^2 - p_y^2)(2q_x + p_y)y + \{2(q_x + 2p_y)q_{xx} - (p_y + 5q_x)p_{xy}\}z];$$

the polar of $z = 0$ is also the line $\mathfrak{L}$.

For the Darboux tangents we obtain similarly four other covariant lines determined by the quadratic cones:

$$(37)\quad (p_y - q_x)(2p_y + q_x)x^2 + 18q_{yy}yz$$
$$- \frac{xz}{p_y + q_x}\{(5p_y + q_x)q_{xy} - 2(p_y + 2q_x)p_{yy}\}$$
$$+ \frac{z^2}{(p_y + q_x)^2}[3(p_y + q_x)\{p_{yyy}$$
$$+ 3q_{xyy} + 3q_{yz}(p_y + q_x) - \frac{p_{xxx}}{p_{xx}}\, q_{yy}\}$$
$$- 4(2q_{xy} + p_{yy})^2] = 0,$$

$$(38)\quad (q_x - p_y)(2q_x + p_y)y^2 + 18p_{xx}xz$$
$$- \frac{yz}{p_y + q_x}\{(5q_x + p_y)p_{xy} - 2(q_x + 2p_y)q_{xx}\}$$
$$+ \frac{z^2}{(p_y + q_x)^2}[3(p_y + q_x)\{q_{xxx} + 3p_{xxy} + 3p_{xz}(p_y + q_x) - \frac{q_{yyy}}{q_{yy}}\, p_{xx}\}$$
$$- 4(2p_{xy} + q_{xx})^2] = 0.$$

Another quadratic cone is derived from the pencil of these cones, namely,

$$
\begin{aligned}
(39)\qquad & 324p_{xx}q_{yy}(p_y+q_x)^2z^2 \\
&= [2(p_y{}^2-q_x{}^2)(2p_y+q_x)x+\{2(p_y+2q_x)p_{yy}-(5p_y+q_x)q_{xy}\}z] \\
&\cdot\ [2(q_x{}^2-p_y{}^2)(2q_x+p_y)y+\{2(q_x+2p_y)q_{xx}-(5q_x+p_y)p_{xy}\}z].
\end{aligned}
$$

The line $\mathfrak{L}$ may be defined as the polar of the tangent plane $z=0$ with respect to the cone (39) or the intersection of the polar planes of $y=z=0$ and $x=z=0$ with respect to the cones (37) and (38) respectively.

Further we remark one more relation between the line $\mathfrak{L}$ and the cones K_{12} associated with the Segre tangents: *Let* π_r $(r=1,2,3)$ *be the polar plane of a Segre tangent* S_r *with respect to the cone* K_{12} *associated with the remaining Segre tangents; then the harmonic of the tangent plane* $z=0$ *with respect to the trihedron* $\{\pi_1\pi_2\pi_3\}$ *is the line* $\mathfrak{L}$.

8. As has been pointed out by E. Bortolotti,[8] a line can be defined which is associated with a pencil of quadrics

$$
(40)\qquad z=\tfrac{1}{2}(p_y+q_x)xy+(\alpha x+\beta y+\gamma z)z.
$$

It is given by the equations

$$
(41)\qquad \begin{cases} y=2\,\dfrac{2p_{xy}+q_{xx}-3\alpha(p_y+q_x)}{p_y{}^2-q_x{}^2}\,z, \\ x=2\,\dfrac{2q_{xy}+p_{yy}-3\beta(p_y+q_x)}{q_x{}^2-p_y{}^2}\,z. \end{cases}
$$

Conversely, a pencil of quadrics having at O a contact of the second order with $V_3{}^2$ is determined provided that a certain covariant line through O be given.

A comparison of (26) and (41) gives

$$
(42)\qquad \begin{cases} \alpha=\dfrac{(25p_y+53q_x)p_{xy}+2(4p_y+11q_x)q_{xx}}{36(p_y+q_x)(p_y+2q_x)}, \\ \beta=\dfrac{(25q_x+53p_y)q_{xy}+2(4q_x+11p_y)p_{yy}}{36(p_y+q_x)(q_x+2p_y)}. \end{cases}
$$

Hence *the pencil of quadrics associated with the covariant line* $\mathfrak{L}$ *furnishes a new generalization of the Darboux pencil to a non-holonomic surface.*

NATIONAL UNIVERSITY OF CHEKIANG,
KWEICHOW, CHINA.

[8] Cf. Bortolotti, *loc. cit.*, p. 322.

THE PROJECTIVE THEORY OF SURFACES IN RULED SPACE, I.*

By CHENKUO PA.

Introduction. The object of this paper is to study the projective properties of a surface in ruled space. In this treatment certain asymptotic ruled surfaces play an important role. The usual method for investigating these ruled surfaces requires, in general, some lengthy computation. We can however overcome the difficulty by means of a new process which seems simpler than that we have so far used and is particularly useful in the determination of equations of some geometrical elements associated with these ruled surfaces in the normal coordinates of Fubini.

The contents of this paper are divided into two parts. The first part is devoted to the derivation of some invariant quadric surfaces, invariant cones, and invariant curves projectively connected with a point of a general surface. In the last section of this part we further obtain a new definition for any ray in the second canonical pencil such that it may be considered as the principal line of Bompiani [1] for a pair of curves, suitably chosen on the surface at the point. The notions of the Darboux quadrics and of Bompiani-Lane's principal quadrics [2] have been generalized.

In the second part of this paper the equation of the osculating linear complex of every asymptotic osculating ruled surface is first determined, some new invariant quadrics are then introduced as analogues of the Lie quadric and certain special surfaces are finally defined. Among other things we have improved a theorem of Sullivan.[3] A general treatment of the asymptotic ruled surfaces and other ruled surfaces associated with a point such as the asymptotic chord surfaces [4] will be established in a sequel paper. Furthermore we remark that a certain quadric possessing many remarkable properties is defined in a simple way, and that a new property of the Darboux pencil of quadrics has been added. By means of them we can give a simple characterization to the

* Received June 1, 1942.

[1] E. Bompiani, "Invarianti d'intersezione di due curve schembe," *Rendiconti dei Lincei*, (VI), vol. 14 (1931), pp. 456-461.

[2] E. Bompiani, *loc. cit.*; E. P. Lane, "Invariants of intersection of two curves on a surface," *American Journal of Mathematics*, vol. 54 (1932), pp. 699-706.

[3] C. T. Sullivan, "Properties of surfaces whose asymptotic curves belong to linear complexes," *Transactions of the American Mathematical Society*, Vol. 15 (1914), pp. 167-196.

[4] E. Bompiani, "Sugli elementi di 2° ordine delle curve di una superficie," *Rendiconti dei Lincei*, vol. 9 (1929), pp. 288-294.

surfaces whose asymptotic curves belong to linear complexes, the isothermal-asymptotic surfaces, the surfaces whose Lie quadrics all touch a fixed plane [5] etc.

PART A.

The Asymptotic Ruled Surface Q_u^h or Q_v^h.

Let S be an analytic surface in ordinary space referred to its asymptotic net (u, v). It is well known that the non-homogeneous coordinates x, y, z of a generic point P on the surface are solutions of the following system of differential equations,

$$(1) \qquad \begin{aligned} x_{uu} &= \theta_u x_u = \beta x_v, \\ x_{vv} &= \theta_v x_v + \gamma x_u, \end{aligned}$$

and similar equations for y and z.

The integrability conditions of this system are found to be

$$(2) \qquad \begin{aligned} (\theta_{uv} + \beta\gamma)_u - \theta_u(\theta_{uv} + \beta\gamma) &= (\beta_v + \beta\theta_v)_v - \beta(\gamma_u + 2\gamma\theta_u), \\ (\theta_{uv} + \beta\gamma)_v - \theta_v(\theta_{uv} + \beta\gamma) &= (\gamma_u + \gamma\theta_u)_u - \gamma(\beta_v + \beta\theta_v). \end{aligned}$$

For the sake of convenience we normalize the coordinates so that at P $(u = 0,\ v = 0)$

$$(3) \qquad x = y = z = x_v = z_v = y_u = z_u = y_{uv} = x_{uv} = 0, \quad x_u = y_v = z_{uv} = 1.$$

Then the equation of S in the neighborhood of P can be expanded as follows [6]

$$(4) \qquad \begin{aligned} z = xy - \tfrac{1}{3}(\beta x^3 + \gamma y^3) + \tfrac{1}{12}[&(2\beta\theta_u - \beta_u)x^4 - 4(\beta_v + \beta\theta_v)x^3 y \\ &- 6\theta_{uv}x^2y^2 - 4(\gamma_u + \gamma\theta_u)xy^3 + (2\gamma\theta_v - \gamma_v)y^4] + (5). \end{aligned}$$

Suppose that a curve C_u^h of the surface S passes through P and touches at P the asymptotic tangent t_0 $(u = 0)$; it is evidently given by the expression

$$(5) \qquad u = -(h/2)\gamma_v^2 + (3), \qquad (h = \text{const.}),$$

where we have put $h = 1 - j$, j being the contact invariant of C_u^h and the asymptotic curve $u = 0$ at P.[7]

Consider the asymptotic ruled surface Q_u^h composed of the tangents to

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[5] These surfaces were first studied from a projective point of view by Buchin Su. Cf. his paper "On the surfaces whose Lie quadrics all touch a fixed plane, *Science Reports, Univ. of Chekiang*, vol. 2 (1936), pp. 53-61.

[6] Cf. Fubini et Čech, *Introduction à la Géométrie Projective Différentielle des Surfaces*, Paris (1931), p. 73.

[7] Fubini et Čech, *loc. cit.*, p. 56.

the asymptotic curves $v = \text{const.}$ at the points of the curve C_u^h, and, similarly, the asymptotic ruled surface Q_v^h. A generic point $(\bar{x})$ of Q_u^h has the coordinates

$$\bar{x} = \lambda x_u + x, \text{ etc.} \tag{6}$$

Since $\lambda = 0$ at P, we can easily show that the orders of the neighborhoods of $Q_u^h(Q_v^h)$ and the surface S at P are equal to each other. In what follows we shall give a process for investigating the geometry of these ruled surfaces associated with the point P.

1. The fundamental form of Q_u^h. From (6), we have

$$\bar{x}_\lambda = x_u, \quad \bar{x}_{\lambda\lambda} = 0, \quad \bar{x}_{\lambda v} = x_{uv} + (\theta_u x_u + \beta x_v)\frac{du}{dv},$$

$$\bar{x}_v = (*)x_u + \lambda x_{uv} + (\lambda\beta\frac{du}{dv} + 1)x_v, \quad \bar{x}_{vv} = \bar{A}x_v + \bar{B}x_{uv} + (*)x_u,$$

where the quantities $\bar{A}$, $\bar{B}$ are given by

$$\bar{A} = \lambda(\theta_{uv} + \beta\gamma) + 2\lambda(\beta_v + \beta\theta_v)\frac{du}{dv} + \lambda\left(\frac{du}{dv}\right)^2(\beta_u + \beta\theta_u) + \lambda\beta\frac{d^2u}{dv^2} + \theta_v + \beta\left(\right.$$

$$\bar{B} = \lambda\theta_v + 2\lambda\theta_u\frac{du}{dv} + \lambda\beta\left(\frac{du}{dv}\right)^2 + 2\frac{du}{dv}.$$

Therefore

$$D'' = (\bar{x}_v\bar{x}_\lambda\bar{x}_{\lambda\lambda}) = 0, \qquad D' = (\bar{x}_v\bar{x}_\lambda\bar{x}_{\lambda v}) = (x_v x_u x_{uv}),$$

$$D = (\bar{x}_v\bar{x}_\lambda\bar{x}_{vv}) = [-\lambda\bar{A} + (\lambda\beta\frac{du}{dv} + 1)\bar{B}](x_v x_u x_{uv}),$$

so that the differential equation of the curved asymptotic lines on Q_u^h is

$$\frac{d\lambda}{dv} = \tfrac{1}{2}\lambda^2\left[\theta_{uv} + \beta\gamma + 2\beta_v\frac{du}{dv} + (\beta_u - \beta\theta_u)\left(\frac{du}{dv}\right)^2\right.$$

$$\left. + \beta\frac{d^2u}{dv^2} + \beta\frac{du}{dv}\left(\theta_v - \beta\left(\frac{du}{dv}\right)^2\right)\right] - \lambda\left[\theta_u\frac{du}{dv} + \beta\left(\frac{du}{dv}\right)^2\right] - \frac{du}{dv}. \tag{7}$$

Let "d" denote differentiation along one of these curved asymptotic lines, then

$$\frac{d\bar{x}}{dv} = lx_u + mx_v + \lambda x_{ur},$$

where we have put

$$l = \lambda\theta_u\frac{du}{dv} + \frac{du}{dv} + \frac{d\lambda}{dv}, \qquad m = \lambda\beta\frac{du}{dv} + 1. \tag{8}$$

We denote, for brevity, the quantity $d\bar{x}/dv$ by $\bar{x}_v$, although this notation has been already used in a different sense.

Moreover, we put

$$e = \boldsymbol{S}x_u^2, \quad f = \boldsymbol{S}x_u x_v, \quad g = \boldsymbol{S}x_v^2, \quad r = \boldsymbol{S}x_u x_{uv}, \quad s = \boldsymbol{S}x_v x_{uv}, \quad t = \boldsymbol{S}v_{uv}^2,$$
$$E' = \boldsymbol{S}\bar{x}_\lambda^2 = \boldsymbol{S}x_u^2 = e, \quad F' = \boldsymbol{S}\bar{x}_\lambda \bar{x}_v = le + mf + \lambda r,$$
$$G' = \boldsymbol{S}\bar{x}_v^2 = l^2 e + m^2 g + \lambda^2 t + 2lmf + 2l\lambda r + 2m\lambda s,$$

and introduce a new asymptotic parameter w of $Q_u{}^h$ instead of λ such that at P

$$\left(\frac{\partial \lambda}{\partial w}\right)_0 = 1.$$

On account of (7) it is readily seen that

$$(9) \qquad \left(\frac{\partial^2 \lambda}{\partial w \partial v^2}\right)_0 = 0, \quad \left(\frac{\partial^3 \lambda}{\partial w^2 \partial v}\right) = \theta_{uv} + (1-h)\beta\gamma, \quad \left(\frac{\partial^3 \lambda}{\partial w \partial v^2}\right)_0 = \gamma h \theta_u.$$

Therefore the generic point $(\bar{x})$ of the surface $Q_u{}^h$ satisfies the following differential equations:

$$(10) \qquad \bar{x}_{ww} = \frac{\partial^2 \lambda}{\partial w^2} \frac{1}{\dfrac{\partial \lambda}{\partial w}} \bar{x}_w, \qquad \bar{x}_{vv} = \bar{\theta}_v \bar{x}_v + \gamma \bar{x}_w,$$

where

$$(11) \quad \bar{\theta}_v = \begin{Bmatrix} 22 \\ 2 \end{Bmatrix} = \frac{EG_v + FG_w - 2FF_v}{2H^2}, \quad \bar{\gamma} = \begin{Bmatrix} 22 \\ 1 \end{Bmatrix} = \frac{-FG_v - GG_w + 2GF_v}{2H^2},$$

and

$$(12) \qquad E = \boldsymbol{S}\bar{x}_w^2 = \left(\frac{\partial \lambda}{\partial w}\right)^2 E', \quad F = \boldsymbol{S}\bar{x}_w \bar{x}_v = \left(\frac{\partial \lambda}{\partial w}\right) F', \quad G = \boldsymbol{S}\bar{x}_v^2 = G',$$
$$H^2 = EG - F^2.$$

We notice that [8]

$$\bar{x}_w{}^0 = 1, \quad \bar{y}_w{}^0 = 0, \quad \bar{z}_w{}^0 = 0, \quad \bar{x}_v{}^0 = 0, \quad \bar{y}_v{}^0 = 1, \quad \bar{z}_v{}^0 = 0, \quad \bar{x}_{wv}{}^0 = 0,$$
$$\bar{y}_{wv}{}^0 = 0, \quad \bar{z}_{wv}{}^0 = 1, \quad \bar{x}^0 = \bar{y}^0 = \bar{z}^0 = 0,$$

so that the coordinates of $(\bar{x})$ also satisfy the relations (3) at the point P. That is, when $Q_u{}^h$ is referred to the same coordinate system as S at the point P, its expansions in the neighborhood at P *can be obtained by replacing in* (4) *the quantities* β, γ, β_u, β_v, γ_u, γ_v, θ_u, θ_v *etc. by* $\bar{\beta}$, $\bar{\gamma}$, $\bar{\beta}_w$, $\bar{\beta}_v$, $\bar{\gamma}_w$, $\bar{\gamma}_v$, $\bar{\theta}_w$, $\bar{\theta}_v$ *etc., respectively.* But a simple calculation, gives

$$E^0 = 1, G^0 = 1, F^0 = 0, E_w{}^0 = 2\theta_u, E_v{}^0 = 0, G_w{}^0 = 0, G_v{}^0 = 2\theta_v, F_w{}^0 = 0, F_v{}^0 = \gamma,$$
$$(13) \quad F^0{}_{wv} = 1 + \gamma_u + \gamma\theta_u, F^0{}_{vv} = \gamma_v + \gamma\theta_v + \theta_{uv} + (1-h)\beta\gamma, \quad G^0{}_{ww} = 2,$$
$$G^0{}_{wv} = 2[\theta_{uv} + (1-h)\beta\gamma],$$

---

[8] We put $\bar{x}^0 = (\bar{x})_0$, etc. to represent the value of $\bar{x}$, etc. at P.

where we have taken λ such that

$$\left(\frac{\partial^2\lambda}{\partial w^2}\right)_0 = \theta_u.$$

Thus at P we obtain

$$(14)\quad \bar{\beta} = 0,\ \bar{\gamma} = \gamma,\ \theta_v = \theta_v,\ \bar{\theta}_w = \theta_u,\ \bar{\gamma}_v = \gamma_v,\ \bar{\gamma}_w = \gamma_u,\ \bar{\theta}_{wv} = \theta_{uv} + (1-h)\beta\gamma.$$

2. Certain invariant quadrics associated with Q_u^h and Q_v^h. In this section we shall give an application of the above result to the determination of certain invariant quadrics associated with Q_u^h and Q_v^h such as the asymptotic osculating quadrics, asymptotic chord quadrics [9] and the Moutard quadrics of these surfaces. The equation of the Moutard quadric belonging to a tangent whose equations are $z = 0,\ y - nx = 0$ takes the form,

$$[4(\beta + \gamma_u n^2)^2 + 3n(\beta_u + 4\beta_v n + 4\gamma_u n^3 + \gamma_v n^4)]z^2 + 36n^3(z - xy + \tfrac{1}{2}\theta_{uv}z^2)$$
$$- 12n(\beta - 2\gamma_u n^3)(y + \tfrac{1}{2}\theta_u z)z - 12u^2(\gamma_u n^3 - 2\beta)(x + \tfrac{1}{2}\theta_v z)z = 0.$$

Replacing $\beta,\ \gamma,\ \beta_u,\ \beta_v,\ \gamma_u,\ \gamma_v,\ \theta_u,\ \theta_v,\ \theta_{uv}$ in the above equation by $\bar{\beta},\ \bar{\gamma},\ \bar{\beta}_w,\ \bar{\beta}_v,\ \bar{\gamma}_w,\ \bar{\gamma}_v,\ \bar{\theta}_w,\ \bar{\theta}_v,\ \bar{\theta}_{wv}$, respectively, and then using the relations (14), we immediately obtain the equation of the Moutard quadric of Q_u^h belonging to the tangent in question, namely,

$$(15)\quad [4\gamma^2 n^3 + 3n(4\gamma_u + \gamma_v n)]z^2 + 36[z - xy + \tfrac{1}{2}\{\theta_{uv} + (1-h)\beta\gamma\}z^2]$$
$$+ 24n\gamma(y + \tfrac{1}{2}\theta_u z)z - 12n^2\gamma(x + \tfrac{1}{2}\theta_v z)z = 0.$$

Interchanging x and y; β and γ; u and v; n and $1/n$ in the above equation we have the equation of the Moutard quadric of Q_v^h belonging to the same tangent

$$(15')\quad [4\beta^2 + 3n(4\beta_v n + \beta_u)]z^2 + 36[z - xy + \tfrac{1}{2}\{\theta_{uv} + (1-h)\beta\gamma\}z^2]n^3$$
$$+ 24n^2\beta(x + \tfrac{1}{2}\theta_v z)z - 12n\beta(y + \tfrac{1}{2}\theta_u z)z = 0.$$

For $h = 0$, the corresponding quadric has already been obtained by E. Čech.[10] George Wu [11] has recently determined these equations in a more general form.

If $n \to 0$ in the equation (15), we have the equation of a Darboux quadric D_h. Therefore the quadric D_h in the Darboux pencil is the limit of the Moutard quadric of Q_u^h or Q_v^h. More generally, we have the following theorem:

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[9] Cf. E. Bompiani, *loc. cit.*

[10] Fubini-Čech, *Geometria proiettiva differenziale*, vol. 2, Bologna (1927), pp. 529-533.

[11] George Wu, "Systems of quadrics associated with a point of a surface."

The osculating quadric of a non-developable ruled surface along a generator is the limiting position of the Moutard quadric belonging to a non asymptotic tangent t at a point P on this generator when t tends to the tangent of the curved asymptotic line through P.

One of the asymptotic osculating quadrics associated with the curve $dv - \lambda du = 0$ is given by the equation

$$2\lambda^3(xy - z) + 2\beta\lambda yz - 2\beta\lambda^2 xz + [-\lambda^3(\theta_{uv} + \beta\gamma) + \beta\{\lambda' - \beta - (\frac{\beta_u}{\beta} - \theta_u)\lambda - (2\frac{\beta_v}{\beta} + \theta_v)\lambda^2\}]z^2 = 0,$$

where $\lambda' = d^2v/du^2$ while the other is given by

$$2\left(\frac{du}{dv}\right)^3(xy - z) + 2\gamma\frac{du}{dv}xz - 2\gamma\left(\frac{du}{dv}\right)^2 yz + \left[-\left(\frac{du}{dv}\right)^3 \{\theta_{uv} + (1-h)\beta\gamma\} + \gamma\left\{\frac{d^2u}{dv^2} - \gamma - \left(\frac{\gamma_v}{\gamma} - \theta_v\right)\frac{du}{dv} - (2\frac{\gamma_u}{\gamma} + \theta_u)\left(\frac{du}{dv}\right)^2\right]z^2 = 0.$$

By the method we have so far used there is no difficulty in finding the equation of the asymptotic osculating quadrics of the ruled surface $Q_u{}^h$ associated with the curve C_λ: $dv - \lambda du = 0$ on it.[12] Plainly, one of them is the quadric D_h of Darboux, while the other is given by the equation.[13]

$$(16)\quad 2\left(\frac{du}{dv}\right)^3(xy - z) + 2\gamma\frac{du}{dv}xz - 2\gamma\left(\frac{du}{dv}\right)^2 yz + \left[-\left(\frac{du}{dv}\right)^3\{\theta_{uv} + (2-h)\beta\gamma\} + \gamma\left\{\frac{d^2u}{dv^2} - \gamma - \left(\frac{\gamma_v}{\gamma} - \theta_v\right)\frac{du}{dv} - (2\frac{\gamma_u}{\gamma} + \theta_u)\left(\frac{du}{dv}\right)^2\right\}\right]z^2 = 0.$$

Similarly, we obtain, for the ruled surface $Q_v{}^h$, the quadric

$$(16')\quad 2(xy - z) + 2\beta\left(\frac{du}{dv}\right)^2 yz - 2\beta\frac{du}{dv}xz + \left[-\{\theta_{uv} + (2-h)\beta\gamma\} + \beta\left\{-\frac{d^2u}{dv^2} - \beta\left(\frac{du}{dv}\right)^3 - \left(\frac{\beta_u}{\beta} - \theta_u\right)\left(\frac{du}{dv}\right)^2 - (2\frac{\beta_v}{\beta} + \theta_v)\frac{du}{dv}\right\}\right]z = 0.$$

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[12] Of course, the curve C_λ must be on $Q_n{}^h$, but each quadric depends only on the curvilinear element of the second order of C_λ and $Q_n{}^h$ at P has contact of the second order with S. Therefore this condition is not necessary.

[13] Noticing that $dv(d^2u + \theta_u du^2) - du(d^2v + \theta_v dv^2) = 0$ is intrinsic and $\bar{\theta}_v = \theta_v$,

It is remarkable that *the quadrics* (16) *and* (16′) of the surfaces Q'_u *and* Q'_v *for a curve through* P *are precisely the asymptotic osculating quadrics of* S *for a curve of* S *which has a contact of the second order with the given curve at* P.

Since the residual conic of intersection of these two quadrics (16) and (16′) for $Q_u^{h_1}$ and $Q_v^{h_2}$ lies on the plane π_λ whose equation is

$$\left[-\frac{d^2u}{dv^2}\left\{\gamma+\beta\left(\frac{du}{dv}\right)^3\right\}-\beta^2\left(\frac{du}{dv}\right)^6+\gamma^2+(\gamma_v-\gamma\theta_v)\frac{du}{dv}\right.$$
$$+(2\gamma_u+\gamma\theta_u)\left(\frac{du}{dv}\right)^2-(\beta_u-\beta\theta_u)\left(\frac{du}{dv}\right)^3+\beta\gamma\left(\frac{du}{dv}\right)^3(h_2-h_1)$$
$$\left.-(2\beta_v+\beta\theta_v)\left(\frac{du}{dv}\right)^4\right]z-2\frac{du}{dv}\left(\gamma+\beta\left(\frac{du}{dv}\right)^3\right)x+2\left(\frac{du}{dv}\right)^2\left(\beta\left(\frac{du}{dv}\right)^3+\gamma\right)y=0$$

and the osculating plane P_λ of C_λ at P is given by

$$2\left(\frac{du}{dv}\right)^2y-2\frac{du}{dv}x-\left[\beta\left(\frac{du}{dv}\right)^3-\gamma+\frac{du}{dv}\theta_v-\left(\frac{du}{dv}\right)^2\theta_u-\frac{d^2u}{dv^2}\right]z=0,$$

we may conclude that they coincide when and only when the elements of C_λ at P satisfy the following relation:

$$2\frac{d^2u}{dv^2}\left(\gamma+\beta\left(\frac{du}{dv}\right)^3\right)=\beta\gamma\left(\frac{du}{dv}\right)^3(h_2-h_1)+\left[\gamma_v-\beta_u\left(\frac{du}{dv}\right)^4\right]\frac{du}{dv}$$
$$+2\left(\frac{du}{dv}\right)^2\left[\gamma_u-\beta_v\left(\frac{du}{dv}\right)^2\right].$$

For the sake of convenience we shall call a curve having this property *the associate curve*. Especially when $h_1=h_2$, these curves become pangeodesics.[14] Denoting a plane through P by the equation $u_1x+u_2y+u_3z=0$, we have for the osculating plane of the associate curve at P

$$\rho u_1=-2n,\quad \rho u_2=2n^2,$$
$$\rho u_3=-\beta n^3+\gamma-n\theta_v+n^2\theta_u$$
$$+\frac{1}{2(\gamma+\beta n^3)}\left[\beta\gamma n^3(h_2-h_1)+(\gamma_v-\beta_u n^4)n+2(\gamma_u-\beta_v n^2)n^2\right]$$

where $n=du/dv$. In consequence, this plane envelopes a cone Γ_6 of class 6 (22)

$\bar\theta_w=\theta_u$ at P, we may obtain the required equation (16) by substituting (14) into the corresponding equation of the asymptotic osculating quadric of S at P and changing d^2u/dv^2 into $d^2u/dv^2-\beta(du/dv)^3$, the other quantities being unchanged.

[14] For a new definition of pangeodesics cf. E. P. Lane, "Quadriche aventi per generatrici le tangenti asintotiche in un punto di una superficie," *Rendiconti dei Lincei*, (VI), vol. 5 (1927), pp. 100-104.

when the tangent varies at P. The equation of this cone in Fubini's normal coordinates is

$$(17)\quad 2u_1u_2u_3(\beta u_2^3-\gamma u_1^3)-u_1u_2\left\{\frac{\phi}{2}(\beta u_2^3-2\gamma u_1^3)u_2+\frac{\psi}{2}(2\beta u_2^3-\gamma u_1^3)u_1\right\}$$
$$-(\beta^2u_2^6-\gamma^2u_1^6)-\beta\gamma(h_2-h_1)u_2^3u_1^3=0,$$

where we have placed [15]

$$(18)\quad \phi=2\theta_u-\frac{\beta_u}{\beta}=\frac{\gamma_u}{\gamma}+\theta_u,\quad \psi=2\theta_v-\frac{\gamma_v}{\gamma}=\frac{\beta_v}{\beta}+\theta_v,\quad (\theta=\log\beta\gamma).$$

Evidently, the cone (17) is that of Segre when $h_1=h_2$.

One of the asymptotic chord quadrics associated with a curve of S through P has the equation

$$(19)\quad xy-z-\tfrac{1}{2}\beta\frac{du}{dv}xz+\tfrac{1}{6}\beta\left(\frac{du}{dv}\right)^2yz$$
$$-\tfrac{1}{2}\left\{\theta_{uv}+\beta\gamma+\tfrac{1}{3}(\beta_u-\tfrac{1}{2}\beta\theta_u)\left(\frac{du}{dv}\right)^2+(\beta_v+\tfrac{1}{2}\beta\theta_v)\frac{du}{dv}+\tfrac{1}{2}\beta\frac{d^2u}{dv^2}\right\}z^2=0,$$

and the other the equation

$$(20)\quad xy-z-\tfrac{1}{2}\gamma\frac{dv}{du}yz+\tfrac{1}{6}\gamma\left(\frac{dv}{du}\right)^2xz-\tfrac{1}{2}\left\{\theta_{uv}+\beta\gamma+\tfrac{1}{3}(\gamma_v-\tfrac{1}{2}\gamma\theta_v)\left(\frac{dv}{du}\right)^2\right.$$
$$\left.+(\gamma_u+\tfrac{1}{2}\gamma\theta_u)\frac{dv}{du}-\tfrac{1}{2}\gamma\left(\frac{dv}{du}\right)^3\frac{d^2u}{dv^2}\right\}z^2=0.$$

For the curve $C_u^h(C_v^h)$ the first (second) quadric becomes a Darboux quadric $D_{\frac{1}{2}h}$:

$$(21)\qquad xy-z-\tfrac{1}{2}\{\theta_{uv}+(1-\tfrac{1}{2}h)\}z^2=0.$$

If $C_u^h(C_v^h)$ has a point of inflexion at P, we obtain *the quadric* $D_{\frac{1}{2}}$; if C_u^h has a stationary osculating plane at P, we have *the quadric* $D_{1/6}$.

By the method we have so far used, we find the corresponding quadrics for the surface Q_u^h. In fact, one of them is a quadric of Darboux and the other is given by the equation

$$(22)\quad xy-z-\tfrac{1}{2}\gamma\frac{dv}{du}yz+\tfrac{1}{6}\gamma\left(\frac{dv}{du}\right)^2xz-\tfrac{1}{2}\left\{\theta_{uv}+(\tfrac{3}{2}-h)\beta\gamma+\tfrac{1}{3}(\gamma_v-\tfrac{1}{2}\gamma\theta_v)\left(\frac{dv}{du}\right)^2\right.$$
$$\left.+(\gamma_u+\tfrac{1}{2}\gamma\theta_u)\frac{dv}{du}-\tfrac{1}{2}\gamma\left(\frac{dv}{du}\right)^3\frac{d^2u}{dv^2}\right\}z^2=0.$$

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[15] In fact, if an equation in our coordinate system contains only the quantities $\beta, \gamma, \theta_u, \theta_v, \beta_u, \beta_v, \gamma_u, \gamma_v, \theta_{uv}$, then it can be transformed to Fubini's normal coördinate system by putting $\theta=\log\beta\gamma$ in the equation without changing other quantities.

Similarly, we obtain, for the ruled surface, the quadric

$$(23)\quad xy - z - \tfrac{1}{2}\beta \frac{du}{dv} xz + \tfrac{1}{6}\beta \left(\frac{du}{dv}\right)^2 yz - \tfrac{1}{2} \left\{ \theta_{uv} + (\tfrac{3}{2} - h)\beta\gamma + \tfrac{1}{3}(\beta_u - \tfrac{1}{2}\beta\theta_u)\left(\frac{du}{dv}\right)^2 + (\beta_v + \tfrac{1}{2}\beta\theta_v)\frac{du}{dv} + \tfrac{1}{2}\beta \frac{d^2u}{dv^2} \right\} z^2 = 0.$$

For $h = \frac{1}{2}$, these quadrics are asymptotic chord quadrics of S at P.

3. The asymptotic curves of Q_u^h and Q_v^h. The osculating quadric of Q_u^h (or Q_v^h) at P is the Darboux quadric D_h of S. As any Darboux quadric must pass through the asymptotic tangents of S at P, and Q_u^h (or Q_v^h) has at P, the same tangent plane as S, the two asymptotic directions of Q_u^h (or Q_v^h) at P coincide with those of S at the same point. Consequently, the curved asymptotic line of Q_u^h (or Q_v^h) at P is tangent to the asymptotic curve $u = 0$ (or $v = 0$) of S. We can, however, show a more precise result, namely,

The curved asymptotic line of Q_u^h (Q_v^h) and the asymptotic curve $u = 0$ ($v = 0$) of S have a common osculating linear complex at P.

To prove this, let the Plücker coordinates of a line joining two points (x) and $(\bar{x})$ be denoted by

$$p_{12} = x\bar{y} - \bar{x}y,\ p_{13} = x\bar{z} - \bar{x}z,\ p_{23} = y\bar{z} - \bar{y}z,\ p_{14} = x - \bar{x},\ p_{24} = y - \bar{y},\ p_{34} = z - \bar{z},$$

then the asymptotic osculating linear complex of asymptotic curves $u = 0$ of S at P is given by the following equation:

$$(24)\qquad p_{12} - p_{34} + \left(\theta_u + \frac{\gamma_u}{\gamma}\right) p_{13} = 0.$$

In order to obtain the asymptotic osculating linear complex of the curved asymptotic curve of Q_u^h at P, we have to replace θ_u, γ_u, γ in the above equation by $\bar{\theta}_w = \theta_u$, $\bar{\gamma}_w = \gamma_u$ and $\bar{\gamma} = \gamma$, respectively, which obviously leaves the equation unchanged.

In a similar way we can establish the result for Q_v^h.

We come now to consider the expansion for the curved asymptotic line of Q_u^h (or Q_v^h) at P up to higher orders. For this purpose the fundamental quantities of the surface Q_u^h (or Q_v^h) are required.

Since

$$x_{vvv} = \frac{d^3x}{dv^3} + 3\frac{d^2\lambda}{dv^2}\frac{dx_u}{dv} + 3\frac{d\lambda}{dv}\frac{d^2x_u}{dv^2} + \lambda\frac{d^3x_u}{dv^3} + \frac{d^3\lambda}{dv^3}x_u,$$

and $\bar{\theta}_v^2 + \bar{\theta}_{vv} = \theta_v^2 + \theta_{vv}$, $\bar{\theta}_v = \theta_v$ at P, we have

(25) $$\bar{\theta}_{vv} = \theta_{vv},$$

and

$$(\bar{x}_{vvvv})_0 = \left(\frac{d^3x_v}{dv^3} - 3\,\frac{d^2u}{dv^2}\,x_{uvv}\right)_0 = (x_{vvv})_0,$$

(25) $$\bar{\gamma}_{vv} = \gamma_{vv}.$$

It is well known that the power series expansions for the asymptotic curve $u = 0$ of S in the neighborhood of P are of the form

(27) $$\begin{aligned} x &= \bar{a}y^2 + \bar{b}y^3 + \bar{c}y^4 + (5), \\ z &= \bar{r}y^3 + \bar{s}y^4 + \bar{t}y^5 + (6), \end{aligned}$$

where we have placed

(28) $$\begin{aligned} &\bar{a} = \tfrac{1}{2}\gamma,\ \bar{b} = \tfrac{1}{6}(\gamma_v - 2\gamma\theta_v),\ \bar{c} = \tfrac{1}{24}\{\gamma(-2\theta_{vv} + 6\theta_v^2 + \gamma\theta_u + \gamma_u) + \gamma_{vv} - 5\theta_v\gamma_v\}, \\ &\bar{r} = \tfrac{1}{6}\gamma,\ \bar{s} = \tfrac{1}{2}\bar{b}, \qquad\qquad \bar{t} = \tfrac{3}{5}\bar{c} - \tfrac{1}{60}\gamma(\gamma\theta_u + \gamma_u). \end{aligned}$$

An application of the preceding method to this case gives the result that *the curved asymptotic line of Q_u^h has at P a contact of the fourth order at least with the asymptotic curve $u = 0$ of S.*

More generally, we can prove the following theorem:

If a curve C_u^h has at P a contact of order m with the asymptotic curve $u = 0$ of S, then the curved asymptotic line of the asymptotic ruled surface associated with C_u^h has with the asymptotic curve $u = 0$ of S at P a contact of order $m + 3$ at least.

In fact, denote the n-th derivative of a function x along the curved asymptotic line of Q_u^h by $X^{(n)}$ and that along the asymptotic curve $u = 0$ of S by $X^{[n]}$. As this theorem has been proved in the case $m = 1$, we can put $m = n - 1 \geqq 2$ and from the given condition we have at P

$$\frac{du}{dv} = \frac{d^2u}{dv^2} = \cdots = \frac{d^{n-1}u}{dv^{n-1}} = 0,$$

so that at P:

$$\frac{d\lambda}{dv} = \frac{d^2\lambda}{dv^2} = \cdots = \frac{d^{n-1}\lambda}{dv^{n-1}} = 0, \quad \frac{d^n\lambda}{dv^n} = -\frac{d^nu}{dv^n}, \quad \frac{d^{n+1}\lambda}{dv^{n+1}} = -\frac{d^{n+1}u}{dv^{n+1}}.$$

Since $\bar{x} = x + \lambda x_u$ we have at P

$$\bar{x}^{(n)} = x^{(n)} + (\lambda x_u)^{(n)} = x^{(n)} + \lambda^{(n)} x_u = \left(x_u \frac{du}{dv}\right)^{(n-1)}$$

$$+ x_v{}^{(n-1)} + \lambda^{(n)} x_u = x_u \left(\frac{d^n u}{dv^n} + \lambda^{(n)}\right) + x_v{}^{(n-1)} = x_0{}^{(n-1)} = x_v{}^{[n-1]} = x^{[n]},$$

$$\bar{x}^{(n+1)} = x^{(n+1)} + (\lambda x_u)^{(n+1)} = \left(x_u \frac{du}{dv}\right)^{(n)} + x_v{}^{(n)} + \lambda^{(n+1)} x_u + (n+1)\lambda^{(n)} x_u{}^{(1)}$$

$$= x_u \frac{d^{n+1}u}{dv^{n+1}} + n \frac{d^n u}{dv^n} x_u{}^{(1)} + x_{uv} \frac{d^n u}{dv^n} + x_{vv}{}^{[n-1]} + \lambda^{(n+1)} x_u + (n+1)\lambda^{(n)} x_u{}^{(1)}$$

$$= x_u \left(\frac{d^{n+1}u}{dv^{n+1}} + \lambda^{(n+1)}\right) + x_{uv}(n+1)\left(\frac{d^n u}{dv^n} + \lambda^{(n)}\right) + x^{[n+1]} = x^{(n+1)},$$

$$\bar{x}^{(n+2)} = x^{(n+2)} + (\lambda x_u)^{(n+2)} = \left(x_u \frac{du}{dv}\right)^{(n+1)} + x_v{}^{(n+1)} + \lambda^{(n+2)} x_u$$

$$+ (n+2)\lambda^{(n+1)} x_u{}^{(1)} + \frac{(n+2)(n+1)}{2} \lambda^{(n)} x_u{}^{(2)}$$

$$= x_u \frac{d^{n+2}u}{dv^{n+2}} + (n+1) x_{uv} \frac{d^{n+1}u}{dv^{n+1}} + \frac{(n+1)n}{2} x_u{}^{(2)} \frac{d^n u}{dv^n}$$

$$+ x_{uv} \frac{d^{n+1}u}{dv^{n+1}} + n x_{uvv} \frac{d^n u}{dv^n} + x_{uvv} \frac{d^n u}{dv^n}$$

$$+ x^{[n+2]} + \lambda^{(n+2)} x_u + (n+2)\lambda^{n+1} x_{uv} + \frac{(n+2)(n+1)}{2} \left[x_{uu} \frac{d^2 u}{dv^2} + x_{uvv}\right] \lambda^{(n)}$$

As we have at P

$$\frac{d^{n+2}u}{dv^{n+2}} + \lambda^{(n+2)} = -(n+1)\lambda^{(n)} \frac{d^2 u}{dv^2} \theta_u = (n+1) \frac{d^n u}{dv^n} \frac{d^2 u}{dv^2} \theta_u,$$

it follows that

$$\bar{x}^{(n+2)} = x^{[n+2]} - (n-1)\beta \frac{d^2 u}{dv^2} \frac{d^n u}{dv^n} x_v.$$

For $m \geqq 2$ i. e. $n \geqq 3$, we have obtained

$$\bar{x}^{(n+2)} = x^{[n+2]},$$

which completes the proof.

Further, we can prove the following result

If the asymptotic osculating linear complex of the asymptotic curve $u = 0$ at P is stationary, so also is that of the curved asymptotic line of $Q_u{}^0$ at P.

For at P we have

$$\bar{x}_{vvvv} = x_{uvvv} - \beta\gamma^2 h(h+2),$$

$$\bar{\gamma}_{vv} = \gamma_{uv} - 2\gamma^2 h^2 \beta - \beta\gamma^2, \tag{29}$$

and therefore

$$(29') \qquad -\frac{\partial^2 \log \bar{\gamma}}{\partial u \partial v} = k_1 - 2\beta\gamma h^2,$$

where we have placed $k_1 = \beta\gamma - (\partial^2 \log \gamma/\partial u \partial v)$.

By means of these relations we have simultaneously found a geometrical meaning for the invariant $k_1/\beta\gamma$, and similarly for $h_1/\beta\gamma$, where $h_1 = \beta\gamma - (\partial^2 \log \beta/\partial u \partial v)$.

4. Generalizations of Darboux's pencil of quadrics and Bompiani-Lane's principal quadrics. Since

$$\bar{x}_{ww} = \frac{\partial^2 \lambda}{\partial w^2} x_u, \quad \bar{x}_{wwv} = \frac{\partial^3 \lambda}{\partial v \partial w^2} x_u + \frac{\partial^2 \lambda}{\partial w^2}\left(x_{uu}\frac{du}{dv} + x_{uv}\right),$$

we have at P

$$\begin{aligned}
\bar{x}_{wwvv} &= \frac{\partial^4 \lambda}{\partial v^2 \partial w^2} + \theta_u(-\theta_u\gamma h + \gamma_u + \gamma\theta_u) \\
&= (\theta_{uv} + \beta\gamma)_v + 3\beta_v \frac{d^2u}{dv^2} + \beta\frac{d^3u}{dv^3} + \beta\theta_v\frac{d^2u}{dv^2} - \theta_u{}^2\frac{d^2u}{dv^2} + \theta_u(-\theta_u\gamma h + \gamma_u + \gamma\theta_u) \\
&= \theta_v(\theta_{uv} + \beta\gamma) + (\gamma_u + \gamma\theta_u)_u - \gamma(\beta_v + \beta\theta_v) - 3\beta_v\gamma h \\
&\quad + \beta\frac{d^3u}{dv^3} - \beta\gamma h\theta_v + \theta_u{}^2\gamma h + \theta_u(-\theta_u\gamma h + \gamma_u + \gamma\theta_u).
\end{aligned}$$

But

$$\begin{aligned}
(\bar{x}_{wwvv})_0 = \bar{\gamma}_{ww} &+ \gamma_u\theta_u + \gamma\bar{\theta}_{ww} + \theta_u(\gamma_u + \gamma\theta_u) \\
&+ \theta_v(\theta_{uv} + \beta\gamma) - \beta\gamma h\theta_v + \theta_u(-\theta_u\gamma h + \gamma_u + \gamma\theta_u).
\end{aligned}$$

It follows that at P

$$(30) \qquad \bar{\gamma}_{ww} = \gamma_{uu} - \gamma(\beta_v + \beta\theta_v) - 3\beta_v\gamma h + \beta\frac{d^3u}{dv^3} + \gamma(\theta_{uu} - \bar{\theta}_{ww}),$$

where $\bar{\theta}_{ww} = \partial^3\lambda/\partial w^3$. For the following discussion it is not necessary to choose the value of $\partial^3\lambda/\partial w^3$ at P.

Let $Q = x + \xi x_u + \eta x_v + \zeta x_{uv}$ be any point in space; then the conditions of immovability for Q are as follows:

$$(31) \quad \begin{cases} \dfrac{\partial\xi}{\partial u} = -\theta_u\xi - \beta\gamma H\zeta - 1, & \dfrac{\partial\eta}{\partial u} = -\beta\xi - (\beta_v + \beta\theta_v)\zeta, & \dfrac{\partial\zeta}{\partial u} = -\eta - \theta_u\zeta, \\ \dfrac{\partial\xi}{\partial v} = -\gamma\eta - (\gamma_u + \gamma\theta_u)\zeta, & \dfrac{\partial\eta}{\partial v} = -\theta_v\eta - \beta\gamma H\zeta - 1, & \dfrac{\partial\zeta}{\partial v} = -\xi - \theta_v\zeta. \end{cases}$$

The equation of the osculating quadric of $Q_u{}^h$ at P is

$$\xi\eta - \zeta - \tfrac{1}{2}\bar{\theta}_{ww}\zeta^2 = 0.$$

When this quadric moves along the curved asymptotic line of Q_u^h at P, the characteristic curve is given by the equations

$$(32)\quad \begin{cases} \zeta\eta - \zeta - \frac{1}{2}\bar{\theta}_{wv}\zeta^2 = 0, \\ \bar{\gamma}\eta^2 + \bar{\gamma}\bar{\phi}\eta\zeta + \frac{1}{2}(\bar{\theta}_{wwv} - \bar{\theta}_v\bar{\theta}_{wv})\zeta^2 = 0, \end{cases} \quad (\bar{\gamma}\bar{\phi} = \bar{\gamma}_w + \bar{\gamma}\bar{\theta}_w = \gamma_u + \gamma\theta_u).$$

By virtue of the conditions of integrability it is easily shown that at P

$$\bar{\theta}_{wwv} - \bar{\theta}_v\bar{\theta}_{wv} = (\bar{\gamma}_w + \bar{\gamma}\bar{\theta}_w)_w = \bar{\gamma}_{ww} + \gamma\theta_u + \gamma\bar{\theta}_{ww} \equiv T.$$

Therefore the second equation of (32) can be rewritten in the form

$$(33)\qquad \bar{\gamma}\eta^2 + \bar{\gamma}\bar{\phi}\eta\zeta + \tfrac{1}{2}T\zeta^2 = 0,$$

which shows that the characteristic curve in question decomposes into one of the asymptotic tangents at P and two other straight lines R_1 and R_2 with the following equations:

$$R_1:\quad 1 + \tfrac{1}{2}\theta_{wv}\zeta = k_1\xi, \quad \eta = k_1\zeta;$$
$$R_2:\quad 1 + \tfrac{1}{2}\bar{\theta}_{wv}\zeta = k_2\xi, \quad \eta = k_2\zeta,$$

where

$$(34)\quad k_1 = (-\bar{\gamma}\bar{\phi} + (\bar{\gamma}\bar{\phi} - 2\bar{\gamma}T)^{1/2})/2\gamma, \quad k_2 = (-\bar{\gamma}\bar{\phi} - (\bar{\gamma}\bar{\phi} - 2\bar{\gamma}T)^{1/2})/2\gamma.$$

Making use of the relations (14) and (26) we may also represent R_1 and R_2 by the equations

$$(35)\qquad \begin{aligned} R_1:&\quad 1 + \tfrac{1}{2}[\theta_{uv} + (1-h)\beta\gamma]\zeta = k_1\xi, \quad \eta = k_1\zeta, \\ R_2:&\quad 1 + \tfrac{1}{2}[\theta_{uv} + (1-h)\beta\gamma]\zeta = k_2\xi, \quad \eta = k_2\zeta, \end{aligned}$$

where

$$\frac{k_1}{k_2} = \left(-\phi \pm \left(\phi^2 - \frac{2}{\gamma}(r\phi)_u + 2\beta\psi + 6\beta_v h - \frac{2}{\gamma}\beta\frac{d^3u}{dv^3}\right)^{1/2}\right)/2 \quad (\gamma\phi = \gamma_u +$$

In particular for $h = 0$ and $d^3u/dv^3 = 0$ (35) becomes the equations of the sides of the quadrilateral of Demoulin. We notice that the points of intersection of R_1 and R_2 with the tangent plane of S at P are the flecnode points of Q_u^h. Denoting them by P_1 and P_2, we obtain

$$P_1:\quad \zeta = \eta = 0, \quad \xi = 1/k_1,$$
$$P_2:\quad \zeta = \eta = 0, \quad \xi = 1/k_2.$$

The conjugate point of P with respect to P_1 and P_2 is independent of the quantity h and is therefore a fixed point for all Q_u^h. In fact, it is the point of intersection of the second directrix of Wilczynski with the asymptotic tangent $z = y = 0$. Thus we have proved the following theorem:

The correspondence between the two flecnode points of each Q_u^h at P is an involution on the asymptotic tangent $z = y = 0$, and the double points of it are the point P and the point of intersection of the second directrix with the asymptotic tangent.

When $h = 0$, this coincides with a theorem of Demoulin.[16]

Obviously, the straight lines R_1 and R_2 are the flecnode tangents of Q_u^h at P_1 and P_2 respectively. Now, let S_1 and S_2 be the flecnode tangents of Q_v^h so that R_1, R_2 and S_1, S_2 form a quadrilateral Ω_h in space. Then the line joining the two points of intersection of the diagonal L_1, L_2, respectively, of Ω_h with the tangent plane of S at P is *the second directrix*, and the common intersecting line of L_1 and L_2 through P is the first directrix.[17]

It is readily seen that the associate asymptotic ruled surface Q_u^h of the curve C_u^h defined by the equation

$$\frac{d^3u}{dv^3} = 3\gamma h \frac{\beta_v}{\beta}, \tag{37}$$

has at P the same flecnode points as the asymptotic osculating ruled surface Q_u^0. We now discuss some properties of this kind concerning the curves C_u^h and C_v^h, the latter being defined by interchanging the asymptotic curves u and v.

First we shall determine the power series expansion of this curve C_u^h in the neighborhood of P. A curve C traced on the surface and passing through the point P can be expanded in the form

$$x(u(t), v(t)) = x + A_1 t + \tfrac{1}{2}A_2 t^2 + \tfrac{1}{6}A_3 t^3 + \tfrac{1}{24}A_4 t^4 + (5);$$

similar equations hold for $y(u(t), v(t))$, and $z(u(t), v(t))$. When the curve C is not tangent to the asymptotic tangent $v = 0$ we may take v for the parameter t so that

$$A_1 = x_u \frac{du}{dv} + x_v,$$

$$A_2 = x_{uu}\left(\frac{du}{dv}\right)^2 + 2x_{uv}\frac{du}{dv} + x_{vv} + x_u \frac{d^2u}{dv^2},$$

$$A_3 = x_{uuu}\left(\frac{du}{dv}\right)^3 + 3x_{uuv}\left(\frac{du}{dv}\right)^2 + 3x_{uvv}\frac{du}{dv} + x_{vvv} + 3\left[x_{uu}\frac{du}{dv}\frac{d^2u}{dv^2} + x_{uv}\frac{d^2u}{dv^2}\right] + x_u \frac{d^3u}{dv^3},$$

. . . . . .

[16] A. Demoulin, "Sur la quadrique de Lie," *Comptes Rendus*, vol. 147 (1908), pp. 493-496.

[17] For the quadrilateral of Demoulin the result has been pointed out by Bompiani and Godeaux.

On account of the relations (3), we obtain the required expansions for the curve C_u^h, namely,

$$(38)\qquad \begin{cases} x = \frac{\gamma}{2}(1-h)v^2 + \frac{1}{6}\left[\gamma_v + \gamma\theta_v + \frac{d^3u}{dv^3}\right]v^3 + (4), \\ y = v + \frac{1}{2}\theta_v v^2 + \frac{1}{6}(\theta_{vv} + \theta_v^2)v^3 + (4), \\ z = \frac{\gamma}{6}(1-3h)v^3 + \frac{1}{12}\left[\gamma\theta_v(1-3h) + \gamma_v + 2\frac{d^3u}{dv^3}\right]v^4 + (5), \end{cases}$$

or

$$(38')\qquad \begin{cases} x = \frac{\gamma}{2}(1-h)y^2 + \left[-\frac{\gamma}{2}(1-h)\theta_v + \frac{1}{6}\left(\gamma_v + \gamma\theta_v + \frac{d^3u}{dv^3}\right)\right]y^3 + (4), \\ z = \frac{\gamma}{6}(1-3h)y^3 + \left[\frac{1}{12}\left\{\gamma\theta_v(1-3h) + \gamma_v + 2\frac{d^3u}{dv^3}\right\} - \frac{1}{4}\gamma\theta_v(1-3h)\right]y^4 + (5). \end{cases}$$

When $h = 1$ this curve has a point of inflexion at P; when $h = \frac{1}{3}$ the osculating plane at P is stationary; when $h = 0$ this curve has a contact of the second order with the asymptotic curve $v = 0$ at P, and when $h = 2$ this curve has a harmonic contact with $u = 0$ at P. It is easy to show that for any pair of curves C_u^h and C_v^h *the three principal points of Bompiani*[18] *are collinear.* In order to determine the line on which lie the principal points, we rewrite the expansions (38′) for the curve (37) in the normal coordinate system of Fubini:

$$(39)\qquad \begin{cases} x = \frac{\gamma}{2}(1-h)y^2 + \frac{\gamma(3h-1)}{6}\frac{\partial \log \beta^2\gamma}{\partial v}y^3 + (4), \\ z = \frac{\gamma}{6}(1-3h)y^3 + \frac{\gamma(6h-1)}{12}\frac{\partial \log \beta^2\gamma}{\partial v}y^4 + (5). \end{cases}$$

Interchanging the pairs x, y; β, γ; u, v we have the expansions for the curve C_v^h. The line under consideration is given by the equation

$$(40)\qquad \frac{1-6h}{4(1-3h)}\frac{\partial \log \beta\gamma^2}{\partial u}x + \frac{1-6h}{4(1-3h)}\frac{\partial \log \beta^2\gamma}{\partial v}y + 1 = 0.$$

Hence we obtain the following theorem:

The principal points for any pair of curves C_u^h and C_v^h lie on a canonical line l_h of the second species. In particular, for the four cases we have so far quoted, the lines are

L_0: The edge of Green;[19]
$L_{1/3}$: The canonical tangent;
L_1: The line $C(-\frac{5}{8})$;
L_2: The canonical line $C(-\frac{11}{20})$.[20]

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[18] E. Bompiani, *loc. cit.* [19] Cf. E. Bompiani, *loc. cit.*

[20] Cf. Buchin Su, "Osculating conics of the plane sections through a point of a surface," *American Journal of Mathematics*, vol. 65 (1943), pp. 439-449.

The quadric which has at P a contact of the fourth order with both C_u^h and C_v^h can be determined as follows:

The equation of this quadric must necessarily be of the form

$$xy + k_1 z + k_2 xz + k_3 yz + k_4 z^2 = 0. \tag{40}$$

Substituting the expansions (39) into (41), and demanding that the coefficients of y^0, y, y^2, y^3, y^4 in the result must be equal to zero, we have

$$\frac{\gamma}{2}(1-h) - \frac{\gamma}{6}(1-3h)k_1 = 0, \quad \frac{\gamma}{6}(1-3h)k_3 + \frac{\gamma(3h-1)}{6}\psi + \frac{\gamma(6h-1)}{12}k_1\psi = 0.$$

Similarly we obtain the conditions for another asymptotic curve:

$$\frac{\beta}{2}(1-h) - \frac{\beta}{6}(1-3h)k_1 = 0, \quad \frac{\beta}{6}(1-3h)k_2 + \frac{\beta(3h-1)}{6}\phi + \frac{\beta(6h-1)}{12}k_1\phi = 0.$$

Therefore there exist ∞^1 quadrics in question, each of them being given by the equation

$$xy - \frac{3(1-h)}{1-3h}z + \frac{9h-1}{2(1-3h)^2}\phi xz + \frac{9h-1}{2(1-3h)^2}\psi yz + kz^2 = 0. \tag{42}$$

When $h = 0$ the corresponding pencil consists of the principal quadrics.[21] If the curve C_u^h and C_v^h have P for a point of inflexion, the corresponding quadrics belong to a pencil of cones:

$$xy + \phi xz + \psi yz + kz^2 = 0. \tag{43}$$

The polar line of the tangent plane at P with respect each of these cones is the first canonical line $C(-1)$.[22]

The common polar reciprocals of the quadrics (42) and the quadrics of Darboux are the line

$$\frac{1-9h}{4(1-3h)}\phi x + \frac{1-9h}{4(1-3h)}\psi y + 1 = 0, \tag{44}$$

and its conjugate with respect to any quadric of Darboux. In particular when $h = 0$ the corresponding lines are the edges of Green.[23]

Consider the linear complexes each of which has at P a contact of the third order with C_u^h; they are given by the following equation:

$$(1-3h)^2 p_{12} - h(1+3h)\psi p_{23} - (1-h)(1-3h)p_{34} + kp_{13} = 0, \tag{45}$$

where k is an arbitrary constant.

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[21] Cf. E. P. Lane, *loc. cit.* [22] Cf. E. Bompiani, *loc. cit.* [23] E. P. Lane, *loc. cit.*

Interchanging the indices 1, 2 and ϕ, ψ we have the corresponding linear complexes of C_v^h at P, namely,

$$(46)\quad (1-3h)^2 p_{12} + h(1+3h)\phi p_{13} + (1-h)(1-3h)p_{34} + k' p_{23} = 0,$$

k' being an arbitrary constant.

We come now to a generalization of the notion of the Darboux pencil. As any quadric passing through the asymptotic tangents at P may be represented by an equation of the form (41), the Plücker coordinates of a generator L_1 in one regulus are [24]

$$p_{12} = k_1(k_1 + k_3\eta), \quad p_{13} = 0, \quad p_{23} = k_1\eta,$$

$$p_{14} = \frac{k_1}{\eta}(k_1 + k_3\eta), \quad p_{24} = k_2(k_1 + k_3\eta) - k_4\eta, \quad p_{34} = -k_1.$$

and those of L_2 in the other regulus are obtained by interchanging the indices 1, 2 and the pairs of values k_2, k_3; ξ, η.

It can be shown that there exist ∞^1 quadrics passing through the asymptotic tangents such that each regulus belongs to one of the linear complexes (45) and (46) respectively. Each of them is found to be

$$(47)\quad xy - \frac{1-h}{1-3h}z + h\,\frac{1+3h}{(1-3h)^2}\phi xz + h\,\frac{1+3h}{(1-3h)^2}\psi yz + kz^2 = 0,$$

where k denotes a parameter. In particular, when $h = 0$, we obtain the Darboux pencil of quadrics. This generalizes the definition of the Lie quadric, as was shown by Calapso.[25] Moreover, we obtain the pencil of cones (43) by putting $h = 1$.

The common polar reciprocals with respect to the pencil of quadrics (47) and the quadric of Lie are the line

$$(48)\qquad -\frac{1+3h}{2(1-3h)}\phi x - \frac{1+3h}{2(1-3h)}\psi y + 1 = 0, \quad z = 0,$$

and its conjugate. Especially we obtain

The canonical line $C(\frac{1}{2})$, if $h = 0$,
The edge of Green, if $h = 1$,
The canonical line $C(-\frac{7}{10})$, if $h = 2$.

We can further prove the following theorem:

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[24] Compare my paper: "An analogue of Darboux pencil of quadrics," *Science Record, Academia Sinica*, vol. 1 (1942), no. 1.

[25] R. Calapso, "Sui sistemi nulli osculatori alle asintotiche di una data superficie," *Rendiconti dei Lincei* (VI), vol. 13 (1931), pp. 183-185.

The linear complexes which have at P a contact of the fourth order with the curve C_u^h $(h \neq 0)$ defined by (5) and (37) and the asymptotic curve $u = 0$, respectively, have common lines in two planes, namely, the tangent plane and a plane through the first canonical line $c\left(\frac{1+3h}{2(1-3h)}\right)$.

In fact, the common lines of these linear complexes also belong to the following three linear complexes:

$$\begin{cases} (1-3h)^2(x\bar{y}-\bar{x}y)+h\phi(1+3h)(x\bar{z}-\bar{x}z)+(1-h)(1-3h)(z-\bar{z})=0, \\ y\bar{z}-z\bar{y}=0, \\ x\bar{y}-\bar{x}y-z+\bar{z}=0. \end{cases}$$

Let Δ be the determinant formed by the coefficients of $\bar{x}$, $\bar{y}$, $\bar{z}$ in these equations; $\Delta = 0$ gives the required plane $z = 0$ and

$$y - \frac{1+3h}{2(1-3h)}\phi z = 0. \tag{49}$$

A similar result is obtained on combining the curve C_v^h and the asymptotic curve $v = 0$. In this case we obtain the plane

$$x - \frac{1+3h}{2(1-3h)}\psi z = 0. \tag{50}$$

In consequence, the lines of the first canonical pencil are completely determined.

On the contrary, if we combine the linear complex of C_u^h in question with the asymptotic osculating linear complex of the asymptotic curve $v = 0$ at P, there is obtained a similar result. The canonical line in this case is $c\left(-\frac{6h^2-3h+1}{2(1-3h)(1-2h)}\right)$.

PART B.

The Osculating Linear Complexes Associated with the Asymptotic Ruled Surface.

We now proceed to consider the equation of the osculating linear complex of the asymptotic osculating, ruled surface Q_u^0 (or Q_v^0) along the generator through the point P, that is, the linear complex which has at P a contact of the fifth order with Q_u^0 along the generator.

1. The osculating linear complexes. It is of some interest to determine the equation of the required linear complex by the method stated in **4** of Part A. We shall however give a direct computation. For this purpose, we first notice the following results

$$(p_{12})_0 = 0, \quad (p_{13})_0 = 0, \quad (p_{23})_0 = 0, \quad (p_{14})_0 = 0, \quad (p_{24})_0 = -1, \quad (p_{34})_0 = 0;$$

$$\left(\frac{\partial p_{12}}{\partial u}\right)_0 = 1, \quad \left(\frac{\partial p_{13}}{\partial u}\right)_0 = 0, \quad \left(\frac{\partial p_{23}}{\partial u}\right)_0 = 0, \quad \left(\frac{\partial p_{14}}{\partial u}\right)_0 = 0, \quad \left(\frac{\partial p_{24}}{\partial u}\right)_0 = 0, \quad \left(\frac{\partial p_{34}}{\partial u}\right)_0 = -1;$$

$$\left(\frac{\partial^2 p_{12}}{\partial u^2}\right)_0 = \theta_u, \quad \left(\frac{\partial^2 p_{13}}{\partial u^2}\right)_0 = 2, \quad \left(\frac{\partial^2 p_{23}}{\partial u^2}\right)_0 = 0, \quad \left(\frac{\partial^2 p_{14}}{\partial u^2}\right)_0 = -(\theta_{u0} + \beta\gamma), \quad \left(\frac{\partial^2 p_{34}}{\partial u^2}\right)_0 = -\theta_v;$$

$$\left(\frac{\partial^3 p_{12}}{\partial u^3}\right)_0 = \theta_{uu} + \theta_u{}^2 + 3(\beta_v + \beta\theta_v), \quad \left(\frac{\partial^3 p_{13}}{\partial u^3}\right)_0 = 6\theta_u,$$

$$\left(\frac{\partial^3 p_{23}}{\partial u^3}\right)_0 = 2\beta, \quad \left(\frac{\partial^3 p_{14}}{\partial u^3}\right)_0 = -(\theta_{uv} + \beta\gamma)_u - 2\theta_u(\theta_{uv}\beta\gamma),$$

$$\left(\frac{\partial^3 p_{34}}{\partial u^3}\right)_0 = -\theta_{uu} - \beta_v - \beta\theta_v - \theta_u{}^2; \quad \left(\frac{\partial^4 p_{13}}{\partial u^4}\right)_0 = 8\theta_{uu} + 14\theta_u{}^2 + 4(\beta_v + \beta\theta_v),$$

$$\left(\frac{\partial^4 (p_{12} + p_{34})}{\partial u^4}\right)_0 = 2\beta_{uv} + 2\beta_u\theta_v - 2\beta^2\gamma + 8\theta_u\beta_v + 8\beta\theta_u\theta_v,$$

$$\left(\frac{\partial^4 p_{14}}{\partial u^4}\right)_0 = -(\theta_{uv} + \beta\gamma)(3\theta_{uu} + 3\theta_u{}^2 + \beta_v + \beta\theta_v) - (\theta_{uv} + \beta\gamma)_{uu} - 3\theta_u(\theta_{uv} + \beta\gamma)_u.$$

Substituting these values into the determinant

$$\begin{vmatrix} p_{12} & (p_{12})_0 & \left(\frac{\partial p_{12}}{\partial u}\right)_0 & \left(\frac{\partial^2 p_{12}}{\partial u^2}\right)_0 & \left(\frac{\partial^3 p_{12}}{\partial u^3}\right)_0 & \left(\frac{\partial^4 p_{12}}{\partial u^4}\right)_0 \end{vmatrix} = 0,$$

and reducing, we obtain

$$\begin{vmatrix} p_{12} + p_{34} & p_{23} & -2p_{14} - (\theta_{uv} + \beta\gamma)p_{13} \\ \beta_v + \beta\theta_v & \beta & (\theta_{uv} + \beta\gamma)_u - \theta_u(\theta_{uv} + \beta\gamma) \\ \beta_{uv} + \beta_u\theta_v - \beta^2\gamma & \beta_u & (\theta_{uv} + \beta\gamma)_{uu} - \theta_u(\theta_{uv} + \beta\gamma)_u \\ & & -\theta_{uu}(\theta_{uv} + \beta\gamma) - (\theta_{uv} + \beta\gamma)(\beta_v + \beta\theta_v) \end{vmatrix} = 0.$$

Put

$$\beta\gamma H = \theta_{uv} + \beta\gamma, \quad \beta_v + \beta\theta_v = \beta\psi, \quad \gamma_u + \gamma\theta_u = \gamma\phi, \tag{51}$$

$$\beta\gamma - \frac{\partial^2 \log \beta}{\partial u \partial v} = h_1, \qquad \beta\gamma - \frac{\partial^2 \log \gamma}{\partial u \partial v} = k_1, \tag{52}$$

so that

$$\beta\gamma H = h_1 + \psi_u = k_1 + \phi_v, \tag{53}$$

$$(\beta\psi)_{uv} - \beta(\gamma\phi)_u \beta\psi(K + \psi_u) = \beta_{uvv} + \beta_u\theta_{vv} + \beta_{uv}\theta_v - 3\beta\gamma\beta_v - \beta^2\gamma\theta_u - \beta^2\gamma;$$

the required equation of the linear complex L_1 is

$$\left(\psi + \frac{\partial \log \beta h_1}{\partial v}\right)(p_{12} + p_{34} - \psi p_{23}) + (\beta\psi)_v p_{23} + \beta(2p_{14} + \beta\gamma H p_{13} - \gamma\phi p_{23}) = 0. \tag{54}$$

By interchanging the pairs of quantities β, γ; u, v; h_1, k_1 and the indices 1, 2, we obtain the corresponding linear complex L_2 of $Q_v{}^h$ at P, namely,

$$(55)\quad \left(\phi + \frac{\partial \log \gamma k_1}{\partial u}\right)(p_{34} - p_{12}\phi p_{13}) + (\gamma\phi)_u p_{13} + \gamma(2p_{24} + \beta\gamma H p_{23} - \beta\psi p_{13}) = 0.$$

2. A theorem on surfaces whose asymptotic curves belong to linear complexes. If a non-developable ruled surface S be not a quadric and be defined by the equation $\beta = 0$ $(\gamma \neq 0)$, then the osculating linear complex L_1 is indeterminate, while the other linear complex L_2 is given by the equation

$$(55')\quad \left(\phi + \frac{\partial \log \gamma k_1}{\partial u}\right)(p_{34} - p_{12} - \phi p_{13}) + (\gamma\phi)_u p_{13} + \gamma(2p_{24} + \theta_{uv} p_{23}) = 0,$$

where we have put

$$\phi = \frac{\gamma_u}{\gamma} + \theta_u.$$

If one curved asymptotic line $(u = u_0)$ of this ruled surface belongs to a linear complex, then $k_1 = 0$ at $u = u_0$, and the quantity $\partial k_1/\partial u = 0$ or $\neq 0$ according as all the asymptotic curves belong to linear complexes or not. In the latter case, the osculating linear complex of this ruled surface is stationary, and coincides with the osculating linear complex of the asymptotic curve $u = u_0$. In the former case where all curved asymptotic lines belong to linear complexes $(k_1 = 0,\ \partial k_1/\partial u = 0)$, the osculating linear complexes are indeterminate. Thus we have the following theorem:

If a curved asymptotic line on a non-developable ruled surface belongs to a linear complex, then the ruled surface must belong to a linear complex.

We can now improve a theorem of Sullivan [26] in the following form:

If the asymptotic curves of a surface of one system $(u = const.)$ belong to linear complexes, then all the curved asymptotic lines on any asymptotic osculating ruled surface Q_u^0 must necessary belong to linear complexes, and conversely.

In fact, if the asymptotic curves $u = \text{const.}$ belong to linear complexes, we have $k_1 = 0$ identically for u and v and it follows from (55) that all the osculating linear complexes of the asymptotic osculating ruled surfaces Q_u^0 are indeterminate. On account of (55'), we conclude that all the curved asymptotic lines of Q_u^0 belong to linear complexes, which completes the proof.

3. Certain invariant quadrics. The purpose of this section is to prove the following theorem:

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[26] C. T. Sullivan, *loc. cit.*

At a point P of a surface there exist ∞^1 tangent quadrics such that both reguli of each quadric belong to the osculating linear complexes (54) *and* (55) *respectively. Each quadric of the family is completely determined by the contact invariant of the quadric and the surface at P. The quadric of Lie is characterized by the contact invariant I.*

To prove this, let us consider any tangent quadric of the surface S at P:

$$(56)\quad (x+\lambda y)(\mu x+y)+l_1 z+l_2(x+\lambda y)z+l_3(\mu x+y)z+l_4 z^2=0.$$

where λ, μ, l_1, l_2, l_3, l_4 are arbitrary constants.

The Plücker coordinate of a generator l_1 or l_2 in each regulus of this quadric are

$$(57)_1\qquad L_1:\begin{cases} p_{12}=l_1(l_1+l_3Y),\quad p_{13}=-l_1\lambda Y,\quad p_{23}=l_1Y,\\ p_{14}=\left(\dfrac{l_1}{Y}+l_3\right)(l_1-\lambda l_2Y)+\lambda l_4Y,\\ p_{24}=\left(\dfrac{l_1}{Y}+l_3\right)(l_2Y-l_1\mu)-l_4Y,\\ p_{34}=l_1(\lambda\mu-1);\end{cases}$$

$$(57)_2\qquad L_2:\begin{cases} p_{12}=-l_1(l_1+l_2X),\quad p_{13}=l_1X,\quad p_{23}=-l_1\mu X,\\ p_{14}=(l_1+l_2X)\left(l_2-\dfrac{l_1\lambda}{X}\right)-l_4X,\\ p_{24}=(l_1+l_2X)\left(\dfrac{l_1}{X}-\mu l_3\right)+\mu X l_4,\\ p_{34}=l_1(\lambda\mu-1),\end{cases}$$

where X and Y denote parameters.

By means of the equations (54), (55), $(57)_1$, and $(57)_2$, we can easily determine the required quadrics. In fact, in order that the quadric (56) have the property in question, it is necessary and sufficient that

$$(58)\qquad \lambda=\mu=0,\qquad l_3=\frac{l_1+1}{2}\left(\psi+\frac{\partial\log\gamma k_1}{\partial u}\right)\equiv\frac{l_1+1}{2}\Psi,$$

$$l_2=\frac{l_1+1}{2}\left(\phi+\frac{\partial\log\beta k_1}{\partial v}\right)\equiv\frac{l_1+1}{2}\Phi,$$

$$l_4=\frac{(l_1+1)(l_2+2)}{2}\Phi\Psi+\frac{l_1}{2}\beta\gamma H,$$

where

$$(58')\qquad \Phi=\phi+\frac{\partial\log\beta h_1}{\partial v},\qquad \Psi=\psi+\frac{\partial\log\gamma k_1}{\partial u}.$$

Therefore the quadrics are given by the equation

$$(59)\quad xy + l_1 z + \frac{l_1+1}{2}\Phi xz + \frac{l_1+1}{2}\Psi yz + \left\{ \frac{(l_1+1)(l_1+2)}{2}\Phi\Psi + \frac{l_1}{2}\beta\gamma H \right\} z^2 = 0.$$

Plainly, $-l_1$ is the contact invariant of this quadric and the surface at P, and on that account we denote this quadric by $Q(l_1)$. Thus the quadric $Q(-1)$ is the quadric of Lie.

There are two distinct quadrics $Q(l_1)$ passing through a generic point $P'(x', y', z')$ in space. They coincide when and only when the point P lies on the following quadric

$$(60)\quad [1 + \tfrac{1}{2}\Phi x + \tfrac{1}{2}\Psi y + \tfrac{1}{2}(\beta\gamma H + 3\Phi\Psi)z]^2 - 2\Phi\Psi[xy + \tfrac{1}{2}\Phi xz + \tfrac{1}{2}\Psi yz + \Phi\Psi z^2] = 0.$$

We note that this quadric Q is the envelope of the quadrics $Q(l_1)$.

4. Some special surfaces and a pencil of invariant lines. Noticing that the discriminant of (60) is

$$\begin{vmatrix} \Phi^2 & -3\Phi\Psi & \Phi(\beta\gamma H + \Phi\Psi) & 2\Phi \\ -3\Phi\Psi & \Psi^2 & \Psi(\beta\gamma H + \Phi\Psi) & 2\Psi \\ \Phi(\beta\gamma H + \Phi\Psi) & \Psi(\beta\gamma H + \Phi\Psi) & (\beta\gamma H + 3\Phi\Psi)^2 - \gamma(\Phi\Psi)^2 & 2(3\Phi\Psi + \beta\gamma H) \\ 2\Phi & 2\Psi & 2(3\Phi\Psi + \beta\gamma H) & 4 \end{vmatrix} = 64(\Phi\Psi)^4,$$

We obtain the following result:

The quadric is never singular unless it decomposes into a double plane.

The equation of this plane is

$$(61)\qquad 1 + \tfrac{1}{2}\Phi x + \tfrac{1}{2}\Psi y + \tfrac{1}{2}(\beta\gamma H + 3\Phi\Psi)z = 0, \qquad (\Phi\Psi = 0),$$

and *the surface is characterized by* $\Phi\Psi = 0$.

Let us now determine a class of surfaces whose quadrics $Q(l_1)$ all touch a fixed plane. In this case the plane (61) must be stationary. Hence the required surfaces are defined by $\Phi = \Psi = H = 0$. Consequently, the surfaces of this class are characterized by the fact that all quadrics of Lie touch a fixed plane [27] and the two projective normals coincide with the two directrices of the two osculating linear complexes (54) and (55). The second projective normals lie in the same fixed plane.

Differentiating (61) with respect to u and using the conditions of immovability (31), we obtain the relations

[27] Buchin Su, *loc. cit.*

$$\frac{\partial\Phi}{\partial u} = \theta_u\Phi + \beta\Psi - \tfrac{1}{2}\Phi^2, \qquad \frac{\partial\Psi}{\partial u} = \beta\gamma H + \tfrac{5}{2}\Phi\Psi,$$

$$\frac{\partial}{\partial u}(\beta\gamma H + 3\Phi\Psi) = \theta_u(\beta\gamma H + 3\Phi\Psi) + \Psi(\beta_v + \beta\theta_v) + \beta\gamma H\Phi - \tfrac{1}{2}\Phi(\beta\gamma H + 3\Phi\Psi).$$

In a similar way we have

$$\frac{\partial\Phi}{\partial v} = \beta\gamma H + \tfrac{5}{2}\Phi\Psi, \qquad \frac{\partial\Psi}{\partial v} = \theta_v\Psi + \gamma\Phi - \tfrac{1}{2}\Psi^2,$$

$$\frac{\partial}{\partial v}(\beta\gamma H + 3\Phi\Psi) = \theta_v(\beta\gamma H + 3\Phi\Psi) + \Phi(\gamma_u + \gamma\theta_u) + \beta\gamma H\Psi - \tfrac{1}{2}\Psi(\beta\gamma H + 3\Phi\Psi).$$

Since $\Phi\Psi = 0$, the unique solution of this system of differential equation is $\Phi = \Psi = H = 0$.

The directrices of the two osculating linear complexes (54) and (55) are given by the equations

$$(62) \qquad \begin{aligned} L_1&: \quad x + \tfrac{1}{2}\Psi z = 0, \quad y + \tfrac{1}{2}\Phi z = 0; \\ L_2&: \quad \tfrac{1}{2}\Phi x + \tfrac{1}{2}\Psi y + 1 = 0, \quad z = 0, \end{aligned}$$

respectively, and are polar reciprocal with respect to the quadric of Lie. For the sake of convenience they are called the first and the second directrices. For the surface $\Phi = \Psi = 0$, these lines coincide with the first and second projective normals. As the plane (61) intersects the tangent plane in the second directrix l_2, we may conclude that if all the quadrics $Q(l)$ for a surface touch a fixed plane, the first and second projective normals coincide with the first and the second directrices l_1 and l_2, respectively, and all the second directrices lie on a fixed plane.

By virtue of the conditions of integrability we have

$$\Phi = \psi + \frac{1}{h_1}\frac{\partial A^2}{\partial u}, \qquad \Psi = \phi + \frac{1}{k_1}\frac{\partial B^2}{\partial u},$$

where

$$A^2 = -\tfrac{1}{4}\left(2M + 2\frac{\beta_{vv}}{\beta} - \frac{{\beta_v}^2}{\beta^2}\right), \qquad B^2 = -\tfrac{1}{4}\left(2L + 2\frac{\gamma_{uu}}{\gamma} - \frac{{\gamma_u}^2}{\gamma^2}\right)$$

and

$$L = \theta_{uu} - \tfrac{1}{2}{\theta_u}^2 - \beta\theta_v - \beta_v, \quad M = \theta_{vv} - \tfrac{1}{2}{\theta_v}^2 - \gamma\theta_u - \gamma_u.$$

If all the directrices of the two osculating linear complexes (54) and (55) coincide with those of Wilczynski, then the surface must satisfy the following conditions:

$$(63) \qquad \frac{\partial A^2}{\partial u} = 0, \qquad \frac{\partial B^2}{\partial v} = 0.$$

It is evident that the surfaces of Demoulin and Godeaux [28] defined by $A = B = 0$ belong to this class.

Now consider the congruence formed by the first directrices l_1; it is conjugate to the surface if and only if

$$\frac{\partial^2 \log \beta^2 h_1}{\partial u \partial v} = \frac{\partial^2 \log \gamma^2 k_1}{\partial u \partial v}. \tag{64}$$

The surfaces of this class may be seen as generalizations of the isothermally asymptotic surfaces.

Let us determine a pencil of invariant lines. By a method used by R. Calapso [29] we make use of the quadric $Q(l_1)$ instead the quadric of Lie. The corresponding line here obtained is given by the equations

$$x + \frac{\psi + (l_1+1)\Psi}{6 + 2l_1} z = 0, \qquad y + \frac{\phi + (l_1+1)\Phi}{6 + 2l_1} z = 0. \tag{65}$$

All these lines corresponding to various values of l_1 lie in the following plane:

$$(2\phi - 4\Phi)x - (2\psi - 4\Psi)y + (\phi\Psi - \psi\Phi)z = 0. \tag{66}$$

For $l = -1$ we obtain the edge of Green. This pencil coincides with the first canonical pencil for the surface (63).

5. Further results on the linear complexes. This section is devoted to the demonstration the following two results:

(i) *A necessary and sufficient condition that a space curve should belong to a linear complex is that all the osculating linear complexes have a straight line in common.*

(ii) *Each regulus of any quadric passing through the quadrilateral of Demoulin of a surface at a point P belongs to one of the two asymptotic osculating linear complexes at P.*

There is no difficulty in proving (i). We proceed, therefore, to establish the result (ii).

To this end we remark that each regulus of the quadric of Lie belongs to one of the two asymptotic osculating linear complexes at the point P.[30] Let

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[28] Cf. L. Godeaux, "Sur les surfaces ayant mêmes quadriques de Lie (Première note), *Academie Royale de Belgique* (5), vol. 14 (1928), pp. 158-173.

[29] R. Calapso, "Un tecrema sullo spigolo di Green," *Rendiconti dei Lincei* (VI), vol. 13 (1931), pp. 266-268.

[30] R. Calapso, *loc. cit.*

us denote the four characteristic points, other than P, on the corresponding quadric of Lie by A, B, C, D so that $\{ABCD\}$ is the quadrilateral of Demoulin, $\overline{AD}, \overline{BC}$ being the diagonals. Let $\mathfrak{L}_1$ and $\mathfrak{L}_2$ be the two asymptotic osculating linear complexes such that $\overline{AC}$, $\overline{BC}$ and $\overline{AB}$, $\overline{CD}$ belong to $\mathfrak{L}_1$ and $\mathfrak{L}_2$, respectively. It is required to prove that the lines p_1 and p_2 passing through a generic point P' in space and intersecting $\overline{AB}$, $\overline{CD}$ and $\overline{AC}$, $\overline{BD}$, respectively, must belong to $\mathfrak{L}_1$ and $\mathfrak{L}_2$, respectively.

As shown by Bompiani [31] and Godeaux,[32] the diagonals $\overline{AD}$, $\overline{BC}$ intersect the directrices of Wilczynski and therefore $\overline{AD}$, $\overline{BC}$ are common lines of $\mathfrak{L}_1$ and $\mathfrak{L}_2$. Denote the point of intersection of p_1 with $\overline{CD}$ by l and the null systems defined by $\mathfrak{L}_1$ and $\mathfrak{L}_2$ by $\{\mathfrak{L}_1\}$ and $\{\mathfrak{L}_2\}$, respectively; then the null plane of A in $\{\mathfrak{L}_1\}$ is the plane ACD and consequently $\overline{AL}$ belongs to $\mathfrak{L}_1$. The null plane of l in $\{\mathfrak{L}_1\}$ is the plane ABL. Hence p_1 belongs to $\mathfrak{L}_1$ and similarly p_2 belongs to $\mathfrak{L}_2$. This proves the theorem.

From the above results follows immediately a theorem of Su.[33] If all the quadrilaterals of Demoulin of a surface lie on a fixed quadric then all the asymptotic curves of the surface must necessarily belong to linear complexes.

THE NATIONAL UNIVERSITY OF CHEKIANG,
KWEICHOW, CHINA.

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[31] E. Bompiani, "Fascio di quadriche di Darboux e normale proiettiva in un punto di una superficie," *Rendiconti dei Lincei* (VI), vol. 6 (1927), pp. 187-190.

[32] L. Godeaux, "Sur les congruences formées par les directrices de Wilczynski d'une surface," *Bull. de l'Acad. Roy. Belgique* (5), vol. 14 (1928), pp. 335-345.

[33] B. Su, "On the surfaces whose asymptotic curves belong to linear complexes," *Tôhoku Mathematical Journal*, vol. 40 (1935), pp. 408-420.

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